

# Quantitative Local and Global A Priori Estimates for Fractional Nonlinear Diffusion Equations

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## Abstract

We establish quantitative estimates for solutions  $u(t, x)$  to the fractional nonlinear diffusion equation,  $\partial_t u + (-\Delta)^s(u^m) = 0$  in the whole range of exponents  $m > 0$ ,  $0 < s < 1$ . The equation is posed in the whole space  $x \in \mathbb{R}^d$ . We first obtain weighted global integral estimates that allow to establish existence of solutions for classes of large data. In the core of the paper we obtain quantitative pointwise lower estimates of the positivity of the solutions, depending only on the norm of the initial data in a certain ball. The estimates take a different form in three exponent ranges: slow diffusion, good range of fast diffusion, and very fast diffusion. Finally, we show existence and uniqueness of initial traces.

**Keywords.** Nonlinear diffusion equation, Fractional Laplacian, Weighted global estimates, Existence for large data, Positivity estimates, Initial trace.

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# 1 Introduction

We consider the class of nonnegative weak solutions of the fractional diffusion equation

$$\partial_t u + (-\Delta)^s(u^m) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \quad (1.1)$$

where  $m > 0$ ,  $0 < s < 1$ ,  $d \geq 1$ , and  $T > 0$ . The precise definition of the fractional Laplacian is given in Appendix 8.1. In most of the paper we assume that the initial data are given:

$$u(0, x) = u_0(x), \quad (1.2)$$

where in principle  $u_0 \in L^1(\mathbb{R}^d)$  and  $u_0 \geq 0$ . However, in Section 2 we consider solutions of possibly changing sign and then we use the notation  $u^m$  to mean  $|u|^{m-1}u$  for powers of signed functions. We refer to [11, 12] for the basic theory of existence and uniqueness of weak solutions for the Cauchy problem (1.1)–(1.2). These papers also comment on the physical motivation and relevance of this nonlocal model, describe the main results on  $L^q$  and  $C^\alpha$  regularity, and give references to related literature. Recently, the existence and properties of Barenblatt solutions for the Cauchy Problem was established in [18]. For  $s = 1$  we recover the classical porous medium/fast diffusion equation, whose theory is well-known, cf. [17]. We will call the case  $s = 1$  the standard diffusion case.

The main purpose of the paper is obtaining quantitative a priori estimates of a local type for the solutions under consideration. Such estimates were obtained for the standard PME by Aronson-Caffarelli [1] and by the authors for the standard FDE [4, 5, 6]. This is not always possible for the present model due to the nonlocal character of the diffusion operator, but then global estimates occur in weighted spaces. The results take different forms according to the value of the exponent  $m$ , a fact that is to be expected since it happens for standard diffusion. The case  $m > 1$  is called the *(fractional) porous medium case*: contrary to the standard porous medium equation, it does not have the property of finite propagation, an important difference established in [11, 12]. The range of exponents  $m \in (0, 1)$  is called the *(fractional) fast diffusion equation*, and it has special properties when  $(d - 2s)/d =: m_c < m < 1$ , which we call the *good fast diffusion range*. When  $0 < m < m_c$  it is known that some solutions extinguish in finite time, which is a clear manifestation of the change of character of the equation, since solutions of the Cauchy problem exist globally in time and are positive everywhere in  $Q = (0, +\infty) \times \mathbb{R}^d$  if  $m \geq m_c$ .

**Outline of the paper and main results.** In Section 2, we derive upper bounds in form of weighted  $L^1$  estimates, valid for nonnegative solutions of the Cauchy problem in the whole fast diffusion range  $0 < m < 1$ . Actually, they are valid for the difference of two ordered solutions, the precise statement is given in Theorem 2.2. Contrary to the purely local  $L^1$  estimates known in the standard fast diffusion case, cf. [10], the estimates for  $s < 1$  are valid in weighted  $L^1$ -spaces and the weight must decay at infinity with a certain decay rate, not too fast, not too slow. This is again a manifestation of the nonlocal properties of the fractional Laplacian. The estimates will be important as a priori bounds for solutions, or families of solutions, through the rest of the paper.

In Section 3 we use the estimates of Section 2 to construct solutions for initial data that belong to weighted  $L^1$ -spaces, in particular for data  $0 \leq u_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$  such that  $u_0(x)$  grows less than  $O(|x|^{2s/m})$  as  $|x| \rightarrow \infty$ , in particular for all bounded data. These solutions can be uniquely identified as minimal solutions in a precise sense and satisfy many of the properties of the known class of bounded and integrable weak solutions.

Section 4 studies the actual positivity of nonnegative solutions via quantitative lower estimates for the good fast diffusion equation. Precise local lower bounds are contained in Theorem 4.1. The behaviour as  $|x| \rightarrow \infty$  (so-called tail behaviour) is studied in Section 4.1, and global spatial lower bounds are derived as a consequence in Section 4.2. The merit of the estimates is that they are quantitative and most of the exponents are sharp. The lower estimates of this section can be adapted for the exponent  $m = m_c$  separating both fast diffusion subranges, but only when  $u_0 \in L^p_{loc}$  for some  $p > 1$ . However, we refrain from doing this particular case in the present paper since the proof uses some other techniques that would lengthen the text.

The very fast diffusion range  $0 < m < m_c$  is studied in Section 5. The weighted  $L^1$  estimates of Theorem 2.2 continue to hold, but this does not allow to obtain the same type of quantitative lower bounds since the technique used in the good fast diffusion range does not work anymore. There are two problems: on the one hand the  $L^1$ – $L^\infty$  smoothing effect does not hold for general  $L^1$  initial data, on the other hand the presence of the extinction phenomenon makes things more complicated, and the extinction time enters directly the estimates of Theorem 5.1. These difficulties have already appeared in the standard FDE,  $s = 1$ , and were treated in our paper [6]. However the technique used in that paper does not extend to  $0 < s < 1$  and we present here a technique that is based on the careful use of weight factors, and in the limit  $s = 1$  gives a simpler proof of the result of [6]. We also study the problem of characterizing the finite time extinction in terms of the initial data; thus, we determine a class of initial data that produces solutions that extinguish in finite time, see Proposition 5.3, as well as a roughly complementary class of initial data for which the solution exists and is positive globally in time, see Corollary 5.2.

Section 6 is devoted to study similar questions for the porous medium case. Theorem 6.1 establishes local lower bounds of the Aronson-Caffarelli type for all  $0 < s \leq 1$ . The question of optimal decay as  $|x| \rightarrow \infty$  is an open problem; for selfsimilar solutions it is solved in [18].

In Section 7 we address a different question that complements our previous results, i.e., the question of existence and uniqueness of an initial trace for nonnegative weak solutions defined in a strip  $Q_T = (0, T] \times \mathbb{R}^d$ . The main results are stated in Theorems 7.2 and 7.3. This result can be combined in the reverse direction with the existence of solutions with initial data a nonnegative Radon measure, Theorem 4.1 of [18]. In the final appendix we collect the definitions of weak, very weak and strong solutions, together with a number of technical results.

We still need to mention the relation of these estimates with the linear fractional heat equation, case  $m = 1$ , for the sake of completeness. The lower bound of Section 6 for  $m > 1$  passes to the limit  $m \downarrow 1$  and solves the first case, and this coincides with the limit  $m \uparrow 1$  of a part of the estimate for  $m < 1$  obtained in Section 4 for  $m < 1$ . See Proposition 4.3.

NOTATIONS. Throughout the paper, we fix  $m_c = (d - 2s)/d$ ,  $m_1 = d/(d + 2s)$ ,  $p_c = d(1 - m)/2$ , and  $\vartheta := 1/[2s - d(1 - m)]$ , which is positive if  $m > m_c$ . We will call  $s$ -Laplacian of  $f$  the function  $-(-\Delta)^s f$ . This is consistent with the use in the standard case  $s = 1$ .

## 2 Weighted $L^1$ estimates in the fast Diffusion range

We will derive weighted  $L^1$  estimates which also hold for the standard FDE (i.e., the limit case  $s = 1$ ). When  $s < 1$  the equation is nonlocal, therefore we cannot expect purely local estimates to hold. Indeed we will obtain estimates in weighted spaces if the weight satisfies certain decay conditions at infinity.

We present first a technical lemma which will be used several times in the rest of the paper.

**Lemma 2.1** *Let  $\varphi \in C^2(\mathbb{R}^d)$  and positive real function that is radially symmetric and decreasing in  $|x| \geq 1$ . Assume also that  $\varphi(x) \leq |x|^{-\alpha}$  and that  $|D^2\varphi(x)| \leq c_0|x|^{-\alpha-2}$ , for some positive constant  $\alpha$  and for  $|x|$  large enough. Then, for all  $|x| \geq |x_0| \gg 1$  we have*

$$|(-\Delta)^s\varphi(x)| \leq \begin{cases} \frac{c_1}{|x|^{\alpha+2s}}, & \text{if } \alpha < d, \\ \frac{c_2 \log |x|}{|x|^{d+2s}}, & \text{if } \alpha = d, \\ \frac{c_3}{|x|^{d+2s}}, & \text{if } \alpha > d, \end{cases} \quad (2.1)$$

with positive constants  $c_1, c_2, c_3 > 0$  that depend only on  $\alpha, s, d$  and  $\|\varphi\|_{C^2(\mathbb{R}^d)}$ . For  $\alpha > d$  the reverse estimate holds from below if  $\varphi \geq 0$ :  $|(-\Delta)^s\varphi(x)| \geq c_4|x|^{-(d+2s)}$  for all  $|x| \geq |x_0| \gg 1$ .

The proof is easy but technical, and is given in Appendix 8.4 for the reader's convenience. We point out that the large-decay case  $\alpha > d$  is what makes the estimate in the fractional Laplacian case very different from the usual Laplacian case. In particular, the  $s$ -Laplacian of a nonnegative smooth function with compact support is strictly positive outside of the support and has a certain decay at infinity, indeed the minimal decay  $|x|^{-(d+2s)}$  is obtained for the  $(-\Delta)^s\varphi$  when  $\varphi \geq 0$  is compactly supported, cf. [12]. A suitable particular choice is the function  $\varphi$  defined for  $\alpha > 0$  as  $\varphi(x) = 1$  for  $|x| \leq 1$  and

$$\varphi(x) = \frac{1}{(1 + (|x|^2 - 1)^4)^{\alpha/8}}, \quad \text{if } |x| \geq 1. \quad (2.2)$$

We are now ready to present the weighted estimates.

**Theorem 2.2 (Weighted  $L^1$  estimates)** *Let  $u \geq v$  be two ordered solutions to the equation (1.1), with  $0 < m < 1$ . Let  $\varphi_R(x) = \varphi(x/R)$  where  $R > 0$  and  $\varphi$  is as in the previous lemma with  $0 \leq \varphi(x) \leq |x|^{-\alpha}$  for  $|x| \gg 1$  and*

$$d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}.$$

*Then, for all  $0 \leq s, t < \infty$  we have*

$$\left( \int_{\mathbb{R}^d} (u(t, x) - v(t, x)) \varphi_R(x) dx \right)^{1-m} \leq \left( \int_{\mathbb{R}^d} (u(s, x) - v(s, x)) \varphi_R(x) dx \right)^{1-m} + \frac{C_1 |t - s|}{R^{2s-d(1-m)}} \quad (2.3)$$

with  $C_1 > 0$  that depends only on  $\alpha, m, d$ .

It is remarkable that the estimate holds for (very) weak solutions, maybe changing sign. Also, it is worth pointing out that the estimate holds both for  $s < t$  and for  $s > t$ . In the limit  $s \rightarrow 1$  we recover the well known  $L^1$  local estimates for the standard FDE.

*Proof.* • STEP 1. A differential inequality for the weighted  $L^1$ -norm. If  $\psi$  is a smooth and sufficiently decaying function we have

$$\begin{aligned}
\left| \frac{d}{dt} \int_{\mathbb{R}^d} (u(t, x) - v(t, x)) \psi(x) dx \right| &= \left| \int_{\mathbb{R}^d} ((-\Delta)^s u^m - (-\Delta)^s v^m) \psi dx \right| \\
&=_{(a)} \left| \int_{\mathbb{R}^d} (u^m - v^m) (-\Delta)^s \psi dx \right| \\
&\leq_{(b)} 2^{1-m} \int_{\mathbb{R}^d} (u - v)^m |(-\Delta)^s \psi| dx \\
&\leq_{(c)} 2 \left( \int_{\mathbb{R}^d} (u - v) \psi dx \right)^m \left( \int_{\mathbb{R}^d} \frac{|(-\Delta)^s \psi|^{\frac{1}{1-m}}}{\psi^{\frac{m}{1-m}}} dx \right)^{1-m}.
\end{aligned}$$

Notice that in (a) we have used the fact that  $(-\Delta)^s$  is a symmetric operator, while in (b) we have used that  $(u^m - v^m) \leq 2^{1-m}(u - v)^m$ , where  $u^m = |u|^{m-1}u$  as mentioned. In (c) we have used Hölder inequality with conjugate exponents  $1/m > 1$  and  $1/(1 - m)$ . If the last integral factor is bounded, then we get

$$\left| \frac{d}{dt} \int_{\mathbb{R}^d} (u(t, x) - v(t, x)) \psi(x) dx \right| \leq C_\psi^{1-m} \left( \int_{\mathbb{R}^d} (u(t, x) - v(t, x)) \psi(x) dx \right)^m$$

Integrating the above differential inequality on  $(s, t)$  with  $s, t \geq 0$  we obtain:

$$\left( \int_{\mathbb{R}^d} (u(t, x) - v(t, x)) \psi(x) dx \right)^{1-m} - \left( \int_{\mathbb{R}^d} (u(s, x) - v(s, x)) \psi(x) dx \right)^{1-m} \leq (1 - m) C_\psi^{1-m} |t - s|$$

which is (2.3) once we estimate the constant  $C_\psi$ , for a convenient choice of test function.

• STEP 2. *Estimating the constant  $C_\psi$ .* Choose  $\psi(x) = \varphi_R(x) := \varphi(x/R) = \varphi(y)$ , with  $\varphi$  as in Lemma 2.1 and  $y = x/R$ , so that  $(-\Delta)^s \psi(x) = (-\Delta)^s \varphi_R(x) = R^{-2s} (-\Delta)^s \varphi(y)$

$$\begin{aligned}
C_\psi &= \int_{\mathbb{R}^d} \frac{|(-\Delta)^s \varphi_R(x)|^{\frac{1}{1-m}}}{\varphi_R(x)^{\frac{m}{1-m}}} dx = R^{d - \frac{2s}{1-m}} \int_{\mathbb{R}^d} \frac{|(-\Delta)^s \varphi(y)|^{\frac{1}{1-m}}}{\varphi(y)^{\frac{m}{1-m}}} dy \\
&= R^{d - \frac{2s}{1-m}} \left[ \int_{B_2} \frac{|(-\Delta)^s \varphi(y)|^{\frac{1}{1-m}}}{\varphi(y)^{\frac{m}{1-m}}} dy + \int_{B_2^c} \frac{|(-\Delta)^s \varphi(y)|^{\frac{1}{1-m}}}{\varphi(y)^{\frac{m}{1-m}}} dy \right] = k_1 R^{d - \frac{2s}{1-m}},
\end{aligned}$$

where it is easy to check that the first integral is bounded, since  $\varphi \geq k_2 > 0$  on  $B_2$ , and when  $|y| > |x_0|$  with  $|x_0| \gg 1$  we know by estimates (2.1) that

$$\frac{|(-\Delta)^s \varphi(y)|^{\frac{1}{1-m}}}{\varphi(y)^{\frac{m}{1-m}}} \leq \begin{cases} \frac{k_3}{|y|^{\alpha + \frac{2s}{1-m}}}, & \text{if } \alpha < d, \\ \frac{k_4 \log |y|}{|y|^{d + \frac{2s}{1-m}}}, & \text{if } \alpha = d, \\ \frac{k_5}{|y|^{\frac{d+2s-\alpha m}{1-m}}}, & \text{if } \alpha > d, \end{cases} \quad (2.4)$$

therefore  $k_1$  is finite whenever  $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$ . Note that all the constants  $k_i$  depend only on  $\alpha, m, d$ .  $\square$

**Remark.** The estimate implies the conservation of mass when  $(d - 2s)/d = m_c < m < 1$ , by letting  $R \rightarrow \infty$ . On the other hand, when  $0 < m < m_c$  solutions corresponding to  $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  with  $p \geq d(1 - m)/2s$ , extinguish in finite time  $T > 0$ , (see e.g. [12]); the above estimates provide a lower bound for the extinction time in such a case, just by letting  $s = T$  and  $t = 0$  in the above estimates:

$$\frac{1}{C_1 R^{d(1-m)-2s}} \left( \int_{\mathbb{R}^d} u_0 \varphi_R dx \right)^{1-m} \leq T \quad (2.5)$$

Moreover, if the initial datum  $u_0$  is such that the limit as  $R \rightarrow +\infty$  of the right-hand side diverges to  $+\infty$ , then the corresponding solution  $u(t, x)$  exists (and is positive) globally in time, as explained in Corollary 5.2.

### 3 Existence of solutions in weighted $L^1$ -spaces

**Theorem 3.1** *Let  $0 < m < 1$  and let  $u_0 \in L^1(\mathbb{R}^d, \varphi dx)$ , where  $\varphi$  is as in Theorem 2.2 with decay at infinity  $|x|^{-\alpha}$ ,  $d - [2s/(1 - m)] < \alpha < d + (2s/m)$ . Then there exists a very weak solution  $u(t, \cdot) \in L^1(\mathbb{R}^d, \varphi dx)$  to equation (1.1) on  $[0, T] \times \mathbb{R}^d$ , in the sense that*

$$\int_0^T \int_{\mathbb{R}^d} u(t, x) \psi_t(t, x) dx dt = \int_0^T \int_{\mathbb{R}^d} u^m(t, x) (-\Delta)^s \psi(t, x) dx dt, \quad \text{for all } \psi \in C_c^\infty([0, T] \times \mathbb{R}^d).$$

*This solution is continuous in the weighted space,  $u \in C([0, T] : L^1(\mathbb{R}^d, \varphi dx))$ .*

*Proof.* Let  $\varphi = \varphi_R$  be as in Theorem 2.2 with the decay at infinity  $|x|^{-\alpha}$ . Let  $0 \leq u_{0,n} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  be a non-decreasing sequence of initial data  $u_{0,n-1} \leq u_{0,n}$ , converging monotonically to  $u_0 \in L^1(\mathbb{R}^d, \varphi dx)$ , i. e., such that  $\int_{\mathbb{R}^d} (u_0 - u_{n,0}) \varphi dx \rightarrow 0$  as  $n \rightarrow \infty$ . Consider the unique solutions  $u_n(t, x)$  of equation (1.1) with initial data  $u_{0,n}$ . By the comparison results of [12] we know that they form a monotone sequence. The weighted estimates (2.3) show that the sequence is bounded in  $L^1(\mathbb{R}^d, \varphi dx)$  uniformly in  $t \in [0, T]$ . By the monotone convergence theorem in  $L^1(\mathbb{R}^d, \varphi dx)$ , we know that the solutions  $u_n(t, x)$  converge monotonically as  $n \rightarrow \infty$  to a function  $u(t, x) \in L^\infty((0, T) : L^1(\mathbb{R}^d, \varphi dx))$ . Indeed, the weighted estimates (2.3) show that when  $u_0 \in L^1(\mathbb{R}^d, \varphi dx)$  then

$$\begin{aligned} \left( \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx \right)^{1-m} &= \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}^d} u_n(t, x) \varphi(x) dx \right)^{1-m} \\ &\leq \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}^d} (u_n(0, x)) \varphi(x) dx \right)^{1-m} + C_1 R^{d(1-m)-2s} t \\ &= \left( \int_{\mathbb{R}^d} u_0(x) \varphi(x) dx \right)^{1-m} + C_1 R^{d(1-m)-2s} t \end{aligned} \quad (3.1)$$

At this point we need to show that the function  $u(t, x)$  constructed as above is a very weak solution to equation (1.1) on  $[0, T] \times \mathbb{R}^d$ , more precisely we have to show that for all  $\psi \in C_c^\infty([0, T] \times \mathbb{R}^d)$  we have

$$\int_0^T \int_{\mathbb{R}^d} u(t, x) \psi_t(t, x) dx dt = \int_0^T \int_{\mathbb{R}^d} u^m(t, x) (-\Delta)^s \psi(t, x) dx dt. \quad (3.2)$$

By the results of [12] we know that each  $u_n$  is a bounded strong solutions, since the initial data  $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , therefore for all  $\psi \in C_c^\infty([0, T] \times \mathbb{R}^d)$  we have

$$\int_0^T \int_{\mathbb{R}^d} u_n(t, x) \psi_t(t, x) dx dt = \int_0^T \int_{\mathbb{R}^d} u_n^m(t, x) (-\Delta)^s \psi(t, x) dx dt. \quad (3.3)$$

Now, for any  $\psi \in C_c^\infty([0, T] \times \mathbb{R}^d)$  we easily have that

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^d} u_n(t, x) \psi_t(t, x) dx = \int_0^T \int_{\mathbb{R}^d} u(t, x) \psi_t(t, x) dx$$

since  $\psi$  is compactly supported and we already know that  $u_n(t, x) \rightarrow u(t, x)$  in  $L_{\text{loc}}^1$ . On the other hand, for any  $\psi \in C_c^\infty([0, T] \times \mathbb{R}^d)$  we have that

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^d} u_n^m(t, x) (-\Delta)^s \psi(t, x) dx dt = \int_0^T \int_{\mathbb{R}^d} u^m(t, x) (-\Delta)^s \psi(t, x) dx dt$$

since  $u_n \leq u$  and

$$\begin{aligned} 0 &\leq \int_0^T \int_{\mathbb{R}^d} (u^m(t, x) - u_n^m(t, x)) (-\Delta)^s \psi(t, x) dx dt \\ &\leq \int_0^T \int_{\mathbb{R}^d} |u(t, x) - u_n(t, x)|^m \varphi^m(x) \frac{|(-\Delta)^s \psi(t, x)|}{\varphi^m(x)} dx dt \\ &\leq \int_0^T \left( \int_{\mathbb{R}^d} |u(t, x) - u_n(t, x)| \varphi(x) dx \right)^m \left( \int_{\mathbb{R}^d} \left| \frac{|(-\Delta)^s \psi(t, x)|}{\varphi(x)^m} \right|^{\frac{1}{1-m}} dx dt \right)^{1-m} \\ &\leq C \int_0^T \int_{\mathbb{R}^d} (u(t, x) - u_n(t, x)) \varphi dx dt \rightarrow 0 \end{aligned}$$

where we have used Hölder inequality with conjugate exponents  $1/m$  and  $1/(1-m)$ , and we notice that

$$\left( \int_{\mathbb{R}^d} \left| \frac{|(-\Delta)^s \psi(t, x)|}{\varphi(x)^m} \right|^{\frac{1}{1-m}} dx dt \right)^{1-m} \leq C$$

since  $\psi$  is compactly supported, therefore by Lemma 2.1 we know that  $|(-\Delta)^s \psi(t, x)| \leq c_3 |x|^{-(d+2s)}$ , and the quotient

$$\left| \frac{|(-\Delta)^s \psi(t, x)|}{\varphi(x)^m} \right|^{\frac{1}{1-m}} \leq \frac{c_3}{|x|^{\frac{d+2s-m\alpha}{1-m}}}$$

is integrable when  $\frac{d+2s-m\alpha}{1-m} > d$  that is when  $\alpha < d + (2s/m)$ . In the last step we already know that  $\int_{\mathbb{R}^d} (u(t, x) - u_n(t, x)) \varphi dx \rightarrow 0$  when  $\varphi$  is as above, i.e. as in Theorem 2.2. Therefore we can let  $n \rightarrow \infty$  in (3.3) and obtain (3.2).

For the solutions constructed above, the weighted estimates (2.3) show that when  $0 \leq u_0 \in L^1(\mathbb{R}^d, \varphi dx)$  imply

$$\left| \int_{\mathbb{R}^d} u(t, x) \varphi_R(x) dx - \int_{\mathbb{R}^d} u(s, x) \varphi_R(x) dx \right| \leq 2^{\frac{1}{1-m}} C_1 R^{d - \frac{2s}{1-m}} |t - s|^{\frac{1}{1-m}} \quad (3.4)$$

which gives the continuity in  $L^1(\mathbb{R}^d, \varphi dx)$ . Therefore, the initial trace of this solution is given by  $u_0 \in L^1(\mathbb{R}^d, \varphi dx)$ .  $\square$

**Remark.** The solutions constructed above only need to be integrable with respect to the weight  $\varphi$ , which has a tail of order less than  $d + 2s/m$ . Therefore, we have proved existence of solutions corresponding to initial data  $u_0$  that can grow at infinity as  $|x|^{(2s/m) - \varepsilon}$  for any  $\varepsilon > 0$ . Note that for the linear case  $m = 1$  this exponent is optimal in view of the representation of solutions in terms of the fundamental solution, but this does not seem to be the case for  $m < 1$ .



**Theorem 3.2 (Uniqueness)** *The solution constructed in Theorem 3.1 by approximation from below is unique. We call it the minimal solution. In this class of solutions the standard comparison result holds, and also the estimates of Theorem 2.2.*

*Proof.* We keep the notations of the proof of Theorem 3.1. Assume that there exist another sequence  $0 \leq v_{0,k} \in L^1(\mathbb{R}^d)$  which is monotonically non-decreasing and converges monotonically to  $u_0 \in L^1(\mathbb{R}^d, \varphi dx)$ . By the same considerations as in the proof of Theorem 3.1, we can show that there exists a solution  $v(t, x) \in C([0, T] : L^1(\mathbb{R}^d, \varphi dx))$ . We want to show that  $u = v$ , where  $u$  is the solution constructed in the same way from the sequence  $u_{0,n}$ . We will prove equality by proving that  $v \leq u$  and then that  $u \leq v$ . To prove that  $v \leq u$  we use the estimates

$$\int_{\mathbb{R}^d} [v_k(t, x) - u_n(t, x)]_+ dx \leq \int_{\mathbb{R}^d} [v_k(0, x) - u_n(0, x)]_+ dx \quad (3.5)$$

which hold for any  $u_n(t, \cdot), v_k(t, \cdot) \in L^1(\mathbb{R}^d)$ , see Theorem 6.2 of [12] for a proof. Letting  $n \rightarrow \infty$  we get that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} [v_k(t, x) - u_n(t, x)]_+ dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} [v_k(0, x) - u_n(0, x)]_+ dx = \int_{\mathbb{R}^d} [v_k(0, x) - u_0(x)]_+ dx = 0$$

since  $v_k(0, x) \leq u_0$  by construction. Therefore also  $v_k(t, x) \leq u(t, x)$  for  $t > 0$ , so that in the limit  $k \rightarrow \infty$  we obtain  $v(t, x) \leq u(t, x)$ . The inequality  $u \leq v$  can be obtained simply by switching the roles of  $u_n$  and  $v_k$ . The validity of estimates of Theorem 2.2 is guaranteed by the above limiting process. The comparison holds by taking the limits in inequality (3.5), as it has been done for  $L^1$ -solutions in [12].  $\square$

## 4 Good fast diffusion range

The first result of the section will be the existence of local lower bounds. In the proof we will use Lemma 8.6, which is a simple optimization lemma that we state in Appendix 8.5. We recall that  $m_c := d/(d - 2s)$  and  $\vartheta := 1/[2s - d(1 - m)]$  which is positive for  $m > m_c$ .

**Theorem 4.1 (Local lower bounds)** *Let  $R_0 > 0$ ,  $m_c < m < 1$  and let  $0 \leq u_0 \in L^1(\mathbb{R}^d, \varphi dx)$ , where  $\varphi$  is as in Theorem 2.2 with decay at infinity  $|x|^{-\alpha}$ ,  $d - [2s/(1 - m)] < \alpha < d + (2s/m)$ . Let  $u(t, \cdot) \in L^1(\mathbb{R}^d, \varphi dx)$  be a very weak solution to Equation (1.1) corresponding to the initial datum  $u_0$ . Then there exists a time*

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{L^1(B_{R_0})}^{1-m} \quad (4.1)$$

such that

$$\inf_{x \in B_{R_0/2}} u(t, x) \geq K_1 R_0^{-\frac{2s}{1-m}} t^{\frac{1}{1-m}} \quad \text{if } 0 \leq t \leq t_*, \quad (4.2)$$

and

$$\inf_{x \in B_{R_0/2}} u(t, x) \geq K_2 \frac{\|u_0\|_{L^1(B_{R_0})}^{2s\vartheta}}{t^{d\vartheta}} \quad \text{if } t \geq t_*. \quad (4.3)$$

The positive constants  $C_*, K_1, K_2$  depend only on  $m, s$  and  $d \geq 1$ .

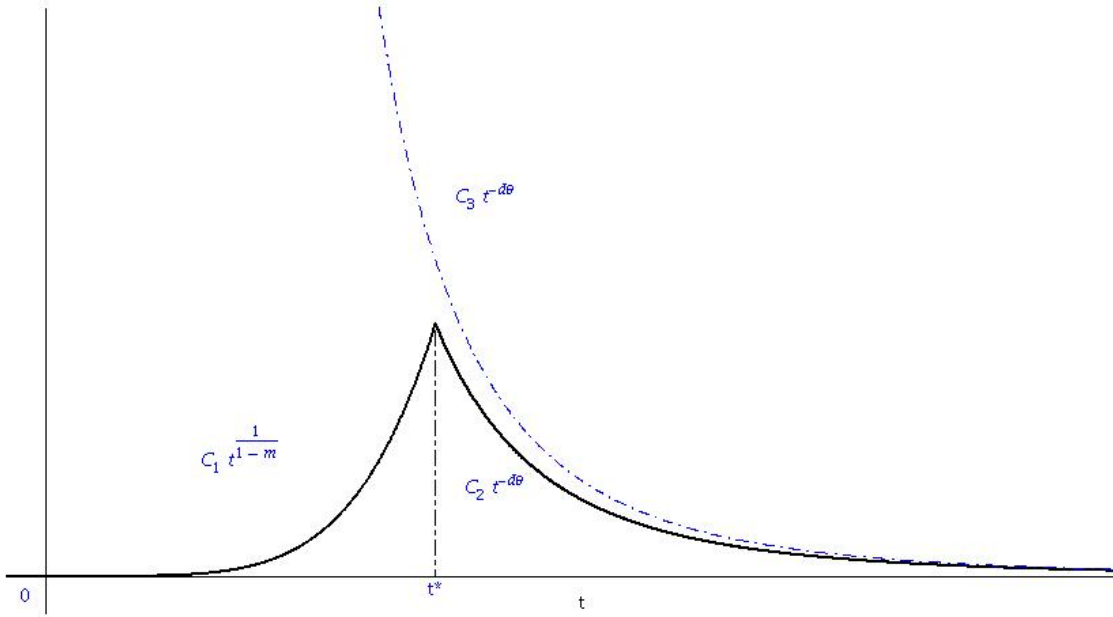


Figure 1: *Black: Lower bounds in the two time ranges. Blue: Upper bounds (smoothing effects), which has the same behaviour when  $t \geq t_*$ .*

**Remarks.** (i) The lower estimate for small times is an absolute bound in the sense that it does not depend on the initial data (though  $t_*$  does depend).

(ii) We obtain the following expressions for  $K_1$  and  $K_2$  and  $C_*$ :

$$\begin{aligned}
 K_1 &:= \frac{K_2}{\left[2^{\frac{2}{\vartheta}+1} s \vartheta (\omega_d I_\infty)^{\frac{1}{d\vartheta}}\right]^{d\vartheta + \frac{1}{1-m}}}, \quad \text{and} \\
 K_2 &:= \left[\left(\frac{2s}{d(1-m)}\right)^{\frac{1}{\vartheta}} - 1\right]^{\frac{1}{1-m}} \left[\frac{d(1-m)}{2s} \frac{(2s\vartheta)^{d(1-m)\vartheta} - 1}{(2s\vartheta)^{d(1-m)\vartheta}}\right]^{\frac{2s\vartheta}{1-m}} \frac{\alpha - d}{2(\alpha - d) + 1} \frac{1}{\omega_d 4^d C_1^{d\vartheta}} \\
 C_* &= 2s\vartheta \left(\omega_d 2^d I_\infty\right)^{\frac{1}{d\vartheta}}
 \end{aligned} \tag{4.4}$$

where  $C_1 > 0$  is the constant in the  $L^1$ -weighted estimates of Proposition 2.2 that depends on  $\alpha, m, d$ , with  $d < \alpha < d + \frac{2s}{m}$ , and  $I_\infty > 0$  is the constant in the smoothing effects (4.6), cf. Theorem 2.2 of [12].

(iii) We can always choose  $\alpha = d/m < d + 2s/m$ , since  $2s > d(1-m)$ .

*Proof.* The proof is divided in several steps.

• **STEP 1. Reduction.** By the comparison principle that it is sufficient to prove lower bounds for solutions  $u$  to the following reduced problem:

$$\begin{cases} \partial_t u + (-\Delta)^s(u^m) = 0, & \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 \chi_{B_{R_0}} = \underline{u}_0, & \text{in } \mathbb{R}^d, \end{cases} \tag{4.5}$$

where  $m_c < m < 1$ ,  $0 < s < 1$ , and  $R_0 > 0$ . We only assume that  $0 \leq u_0 \in L^1(B_{R_0})$ , which implies that  $\underline{u}_0 \in L^1(\mathbb{R}^d)$  since  $\text{supp}(u_0) \subseteq B_{R_0}$  and also that  $\|\underline{u}_0\|_{L^1(\mathbb{R}^d)} = \|u_0\|_{L^1(B_{R_0})}$ . It is not restrictive to assume that the ball  $B_{R_0}$  is centered at the origin.

• **STEP 2. Smoothing effects.** In [12] there are the global  $L^1 - L^\infty$  smoothing effects which provide global upper bounds for solutions to the Cauchy problem 1.1. We apply such smoothing effects to solutions to our reduced Problem 4.5 to get

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{I_\infty}{t^{d\vartheta}} \|u_0\|_{L^1(\mathbb{R}^d)}^{2s\vartheta} = \frac{I_\infty}{t^{d\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} \quad (4.6)$$

where  $\vartheta = 1/[2s - d(1 - m)]$  and the constant  $I_\infty$  only depends on  $d, s, m$ .

• **STEP 3. Aleksandrov principle.** We recall Theorem 11.2 of [18], we have that

$$u(t, 0) \geq u(t, x), \quad \text{for all } t > 0 \text{ and } |x| \geq 2R_0.$$

Therefore one has that

$$\|u(t)\|_{L^\infty(\mathbb{R}^d \setminus B_{2R_0})} = \sup_{x \in \mathbb{R}^d \setminus B_{2R_0}} u(t, x) \leq u(t, 0). \quad (4.7)$$

• **STEP 4. Lower estimates for the  $L^\infty$ -norm on an annulus.** We combine the  $L^1$ -weighted estimates of Theorem 2.2 with the smoothing effects of Step 2: estimates (2.3) read in this context

$$\left( \int_{B_{R_0}} u_0 \, dx \right)^{1-m} \leq \left( \int_{\mathbb{R}^d} u_0 \varphi_R(x) \, dx \right)^{1-m} \leq \left( \int_{\mathbb{R}^d} u(t, x) \varphi_R(x) \, dx \right)^{1-m} + C_1 R^{d(1-m)-2s} t \quad (4.8)$$

we have chosen  $R \geq 2R_0 > 0$  and  $\varphi_R(x) = \varphi(x/R)$  with  $\varphi$  as in Lemma 2.1 (with the explicit form given in formula (2.2)), so that  $\varphi_R(x) = 1$  on  $B_R$  and  $0 \leq \varphi_R(x) \leq |x|^{-\alpha}$  for  $|x| > R$  with  $d - 2s/(1 - m) < \alpha < d + 2s/m$ , and we recall that  $C_1 > 0$  depends only on  $\alpha, m, d$ .

$$\begin{aligned} \|u_0\|_{L^1(B_{R_0})}^{1-m} - C_1 R^{d(1-m)-2s} t &\leq \left( \int_{\mathbb{R}^d} u(t, x) \varphi_R(x) \, dx \right)^{1-m} \\ &\leq \left( \int_{\mathbb{R}^d \setminus B_{2R_0}} u(t, x) \varphi_R(x) \, dx \right)^{1-m} + \left( \int_{B_{2R_0}} u(t, x) \varphi_R(x) \, dx \right)^{1-m} = (I) + (II). \end{aligned} \quad (4.9)$$

We first estimate (I), to this end we observe that if we choose  $d < \alpha < d + 2s/m$  we have that

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_{2R_0}} \varphi_R(x) \, dx &= \int_{\mathbb{R}^d \setminus B_R} \varphi_R(x) \, dx + \int_{B_R \setminus B_{2R_0}} \varphi_R(x) \, dx = \int_{\mathbb{R}^d \setminus B_R} \varphi_R(x) \, dx + \int_{B_R \setminus B_{2R_0}} 1 \, dx \\ &\leq \int_{\mathbb{R}^d \setminus B_R} \frac{1}{[1 + (|x|/R)^2 - 1]^{\alpha/8}} \, dx + \omega_d R^d \\ &= \omega_d R^d \int_1^{+\infty} \frac{r^{d-1}}{[1 + (r^2 - 1)^4]^{\alpha/8}} \, dr + \omega_d R^d \\ &= \omega_d R^d \left[ \int_1^4 \frac{r^{d-1}}{[1 + (r^2 - 1)^4]^{\alpha/8}} \, dr + \int_4^{+\infty} \frac{r^{d-1}}{[1 + (r^2 - 1)^4]^{\alpha/8}} \, dr + 1 \right] \\ &\leq_{(a)} \omega_d R^d \left[ 1 + 4^d + 4^{\alpha/8} \int_4^{+\infty} r^{d-1-\alpha} \, dr \right] = \omega_d R^d \left[ 1 + 4^d + \frac{4^{\alpha/8}}{\alpha - d} \frac{1}{4^{\alpha-d}} \right] \\ &\leq \omega_d 4^d \frac{2(\alpha - d) + 1}{\alpha - d} R^d \end{aligned} \quad (4.10)$$

where we have used that  $R \geq 2R_0$  and in (a) we have used the fact that  $\varphi_R \leq 1$  and that  $1 + (r^2 - 1)^4 \geq r^8/4$ , if  $r \geq 4$ . Therefore we have

$$\begin{aligned} (I)^{\frac{1}{1-m}} &= \int_{\mathbb{R}^d \setminus B_{2R_0}} u(t, x) \varphi_R(x) \, dx \leq \|u(t)\|_{L^\infty(\mathbb{R}^d \setminus B_{2R_0})} \int_{\mathbb{R}^d \setminus B_{2R_0}} \varphi_R(x) \, dx \\ &\leq \omega_d 4^d \frac{2(\alpha - d) + 1}{\alpha - d} R^d \|u(t)\|_{L^\infty(\mathbb{R}^d \setminus B_{2R_0})} \leq \omega_d 4^d \frac{2(\alpha - d) + 1}{\alpha - d} R^d u(t, 0) \end{aligned}$$

where in the last step we have used inequality (4.7) of Step 3, derived from Aleksandrov principle.

We now estimate (II) as follows:

$$\begin{aligned} (II)^{\frac{1}{1-m}} &= \int_{B_{2R_0}} u(t, x) \varphi_R(x) \, dx \leq \|u(t)\|_{L^\infty(\mathbb{R}^d)} \int_{B_{2R_0}} \varphi_R(x) \, dx \\ &\leq_{(a)} \omega_d 2^d R_0^d \|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq_{(b)} \omega_d 2^d R_0^d \frac{I_\infty}{t^{d\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} \end{aligned}$$

where in (a) we have used that  $\varphi_R(x) = 1$  on  $B_R$ ,  $2R_0 < R$  and  $|B_R| = \omega_d R^d$ . In (b) we have used the smoothing effect (4.6). Plugging the above estimates into (4.9) gives

$$\begin{aligned} \|u_0\|_{L^1(B_{R_0})}^{1-m} - C_1 R^{d(1-m)-2s} t &\leq \frac{\left[ \omega_d 2^d R_0^d I_\infty \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} \right]^{1-m}}{t^{d(1-m)\vartheta}} \\ &\quad + \left[ \omega_d 4^d \frac{2(\alpha - d) + 1}{\alpha - d} \right]^{1-m} R^{d(1-m)} u^{1-m}(t, 0), \end{aligned} \quad (4.11)$$

or equivalently

$$\begin{aligned} \left[ \|u_0\|_{L^1(B_{R_0})}^{1-m} - \frac{\left[ \omega_d 2^d R_0^d I_\infty \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} \right]^{1-m}}{t^{d(1-m)\vartheta}} \right] \frac{1}{R^{d(1-m)}} - \frac{C_1 t}{R^{2s}} \\ \leq \left[ \omega_d 4^d \frac{2(\alpha - d) + 1}{\alpha - d} \right]^{1-m} u^{1-m}(t, 0). \end{aligned} \quad (4.12)$$

• **STEP 5. Optimization.** The previous estimate (4.12) is useful only if we can make sure that the left-hand side has a positive lower bound. Let us write inequality (4.12) as

$$F(t, R) := \frac{A(t)}{R^{d(1-m)}} - \frac{B t}{R^{2s}} \leq \left[ \omega_d 4^d \frac{2(\alpha - d) + 1}{\alpha - d} \right]^{1-m} u^{1-m}(t, 0), \quad (4.13)$$

with

$$A(t) = \left[ M - \frac{C}{t^{d(1-m)\vartheta}} \right], \quad M := \|u_0\|_{L^1(B_{R_0})}^{1-m}, \quad C := \left[ \omega_d 2^d R_0^d I_\infty \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} \right]^{1-m}, \quad B = C_1 \quad (4.14)$$

where  $C_1 > 0$  is the constant of  $L^1$ -weighted estimates of Theorem 2.2, and  $I_\infty > 0$  is the constant of the smoothing effects (4.6) of Step 2. We now optimize the function  $F$  as in Lemma 8.6 so that there exists

$$t_* := 2s\vartheta \left( \frac{C}{M} \right)^{\frac{1}{d(1-m)\vartheta}} = 2s\vartheta \left( \omega_d 2^d I_\infty \right)^{\frac{1}{d\vartheta}} R_0^{\frac{1}{\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{1-m} \quad (4.15)$$

and

$$\begin{aligned}\bar{R}(t) &= \left( \frac{2sBt}{d(1-m)A(t)} \right)^\vartheta \geq \bar{R}(t_*) = \left[ \frac{2s}{d(1-m)} \frac{(2s\vartheta)^{2s\vartheta}}{(2s\vartheta)^{d(1-m)} - 1} \right]^\vartheta \frac{B^\vartheta C^{\frac{1}{d(1-m)}}}{M^{\frac{2s\vartheta}{d(1-m)}}} \\ &= \left[ \frac{2s}{d(1-m)} \frac{(2s\vartheta)^{2s\vartheta}}{(2s\vartheta)^{d(1-m)} - 1} \right]^\vartheta \omega_d^{\frac{1}{d}} 2R_0 I_\infty^{\frac{1}{d}} C_1^\vartheta \geq_{(a)} 2R_0,\end{aligned}\tag{4.16}$$

where in (a) we have used that the constants  $I_\infty > 0$  and  $C_1 > 0$  are constants in the upper bounds (4.6) and (4.8) respectively, so that we can chose them to be arbitrarily large to fulfill the condition  $\bar{R}(t_*) \geq 2R$ . Therefore for all  $t \geq t_*$  we have that

$$\begin{aligned}\left[ \omega_d 4^d \frac{2(\alpha-d)+1}{\alpha-d} \right]^{1-m} u^{1-m}(t, 0) &\geq F(\bar{R}(t), t) = \left[ \left( \frac{2s}{d(1-m)} \right)^{\frac{1}{\vartheta}} - 1 \right] \left[ \frac{d(1-m)}{2s} \right]^{2s\vartheta} \frac{A(t)^{2s\vartheta}}{(Bt)^{d(1-m)\vartheta}} \\ &\geq \left[ \left( \frac{2s}{d(1-m)} \right)^{\frac{1}{\vartheta}} - 1 \right] \left[ \frac{d(1-m)}{2s} \right]^{2s\vartheta} \frac{A(t_*)^{2s\vartheta}}{C_1^{d(1-m)\vartheta}} \frac{1}{t^{d(1-m)\vartheta}}\end{aligned}$$

since  $A(t) \geq A(t_*)$  for all  $t \geq t_*$ , and it is easy to check that

$$A(t_*) = \left[ 1 - \frac{1}{(2s\vartheta)^{d(1-m)\vartheta}} \right] \|u_0\|_{L^1(B_{R_0})}^{1-m} = \frac{(2s\vartheta)^{d(1-m)\vartheta} - 1}{(2s\vartheta)^{d(1-m)\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{1-m} > 0,$$

since we recall that  $2s\vartheta > 1$ . Summing up we have obtained

$$u(t, 0) \geq K_2 \frac{\|u_0\|_{L^1(B_{R_0})}^{2s\vartheta}}{t^{d\vartheta}},\tag{4.17}$$

for all  $t \geq t_* > 0$ , where  $K_2$  only depends from  $\alpha, m, s, d$  and takes the form

$$K_2 := \left[ \left( \frac{2s}{d(1-m)} \right)^{\frac{1}{\vartheta}} - 1 \right]^{\frac{1}{1-m}} \left[ \frac{d(1-m)}{2s} \frac{(2s\vartheta)^{d(1-m)\vartheta} - 1}{(2s\vartheta)^{d(1-m)\vartheta}} \right]^{\frac{2s\vartheta}{1-m}} \frac{\alpha-d}{2(\alpha-d)+1} \frac{1}{\omega_d 4^d C_1^{d\vartheta}}\tag{4.18}$$

Note that in the limit  $m \rightarrow 1$  the constant  $K_2 \rightarrow 0$ . By a standard argument it is easy to pass from the center to the infimum on  $B_{R_0/2}(0)$  in the above estimates.

• **STEP 6. Positivity backward in time.** Using Benilan-Crandall estimates which depend only by the homogeneity of the equations, cf. [3]

$$u_t \leq \frac{u}{(1-m)t}\tag{4.19}$$

we can prove positivity in the time interval  $[0, t_*]$ . These estimates in the fractional case has been proven in [12], and imply that the function:  $u(t, x)t^{-1/(1-m)}$  is non-increasing in time, thus for any  $t \in (0, t_*)$  and  $x \in B_{R_0/2}(0)$ , inequality (4.17) gives

$$u(t, x) \geq \frac{u(t_*, x)}{t_*^{\frac{1}{1-m}}} t^{\frac{1}{1-m}} \geq K_2 \frac{\|u_0\|_{L^1(B_{R_0})}^{2s\vartheta}}{t_*^{d\vartheta + \frac{1}{1-m}}} t^{\frac{1}{1-m}} = \frac{K_2}{\left[ 2^{\frac{2}{\vartheta}+1} s\vartheta (\omega_d I_\infty)^{\frac{1}{d\vartheta}} \right]^{d\vartheta + \frac{1}{1-m}}} \left[ \frac{t}{R_0^{2s}} \right]^{\frac{1}{1-m}}$$

where  $K_2 > 0$  is given in (4.18), and  $t_*$  is given by (4.15), and it is easy to check that

$$\frac{\|u_0\|_{L^1(B_{R_0})}^{2s\vartheta}}{t_*^{d\vartheta + \frac{1}{1-m}}} = \frac{\|u_0\|_{L^1(B_{R_0})}^{2s\vartheta}}{\left[2s\vartheta (\omega_d 2^d I_\infty)^{\frac{1}{d\vartheta}} (R_0)^{\frac{1}{\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{1-m}\right]^{d\vartheta + \frac{1}{1-m}}} = \frac{1}{\left[2^{\frac{2}{\vartheta}+1} s\vartheta (\omega_d I_\infty)^{\frac{1}{d\vartheta}}\right]^{d\vartheta + \frac{1}{1-m}} R_0^{\frac{2s}{1-m}}}.$$

The proof is concluded.  $\square$

**Remark.** This lower estimate holds in the limit  $m \rightarrow 1$  and gives lower estimates for the linear fractional heat equation of the form

**Proposition 4.2** *Let  $u \geq 0$  be a weak solution to the Cauchy Problem (1.1), corresponding to  $u_0 \in L^1(\mathbb{R}^d)$  and  $m = 1$ . Then  $\vartheta = 1/2s > 0$  and the estimate says that for given  $R_0$  and  $t_* := C_1 R_0^{2s}$ , then*

$$\inf_{x \in B_{R_0/2}} u(t, x) \geq K_2 \frac{\|u_0\|_{L^1(B_{R_0})}}{t^{d/2s}} \quad \text{if } t \geq t_*. \quad (4.20)$$

The positive constant  $K_2$  depends only on  $C_1$ ,  $s$  and  $d$ .

The proof is easily obtained from the integral representation of the solution.

#### 4.1 Minimal space-like tail behaviour

As a corollary of the previous lower bound, we obtain a quantitative bound from below for the space-like behaviour of any nonnegative solution. We consider a solution that has a certain initial mass  $M$  in the ball of radius 1 and apply the result of Theorem 4.1 after displacing the origin of space coordinates to a point  $x_0$  with  $|x_0| \gg 1$ . We then consider the formula (4.1) for the critical time with center  $x_0$  and radius  $R_0 = |x_0| + 2$ , so that the ball  $B_{R_0}(x_0)$  contains the mass  $M$  mentioned above. As  $R_0 \rightarrow \infty$  also  $t_* \rightarrow \infty$ . We can therefore use the lower bound (4.2) to get an estimate of the form

$$u(t, x_0) \geq G(u_0, t) |x_0|^{-2s/(1-m)}, \quad (4.21)$$

where  $G(u_0, t)$  is given in (4.2). According to the results of [18] the Barenblatt solutions have this precise spatial behaviour in the range  $m_c < m < m_1$ , with  $m_1 = d/(d + 2s)$ , therefore the asymptotic estimate is sharp in this range.

#### 4.2 Global spatial lower bounds in the case $m_1 < m < 1$

We would like to prove that the solution can always be bounded from below by a Barenblatt solution, so the lower bound will be sharp. In the range  $m_c < m < m_1$  the lower bound of Theorem 4.1, gives sharp lower bounds with the same tails as the Barenblatt solutions, as explained in Section 4.1. In the range  $m_1 < m \leq 1$  the lower bound given by (4.21) is not sharp and the following Theorem 4.3 (respectively Proposition 4.2 when  $m = 1$ ) proves that any solution with data in  $L^1(\mathbb{R}^d)$  can always be bounded from below by a Barenblatt solution (respectively by the fundamental solution when  $m = 1$ ). See Figure 2.

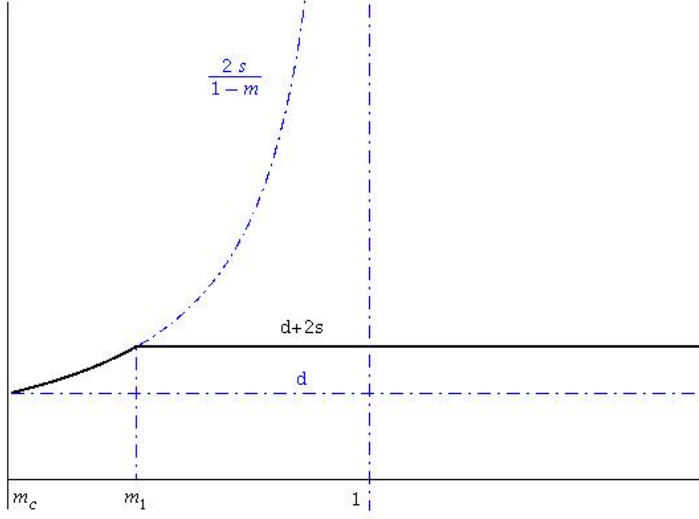


Figure 2: Lower bounds for the spatial decay rates of solutions. Recall that  $m_c = (d - 2s)/d$  and  $m_1 = d/(d + 2s)$

**Theorem 4.3 (Global Lower Bounds when  $m_1 < m < 1$ )** Under the conditions of Theorem 4.1 we have in the range  $m_1 < m < 1$

$$u(t, x) \geq \frac{C(t)}{|x|^{d+2s}} \quad \text{when } |x| \gg 1. \quad (4.22)$$

valid for all  $0 < t < T$  with some bounded function  $C > 0$  that depends on  $t, T$  and on the data.

*Proof.* The proof consists of several steps.

• **STEP 1.** We begin under the extra assumption that  $u_0(x) \geq 2c > 0$  in a ball, that can be taken to be  $B_1(0)$  by scaling. Therefore, there exists a  $t_1$  such that  $u(t, x) \geq c$  for all  $0 \leq t \leq t_1$  and all  $|x| \leq 1$ . We also assume that  $u_0$  is continuous and goes to zero uniformly as  $|x| \rightarrow \infty$ .

Consider the function  $u_{0,\varepsilon}(x) = u_0(x) + \varepsilon$ , and let  $u_\varepsilon(t, x) \geq u(t, x)$  be the corresponding solutions. By the usual theory, [12], we know that  $u_\varepsilon \geq \varepsilon$ ,  $u_\varepsilon - \varepsilon \in L^1(\mathbb{R}^d)$ , since  $u_0 \in L^1(\mathbb{R}^d)$ . Moreover, it is proved in the theory that  $u_\varepsilon \rightarrow \varepsilon$  as  $|x| \rightarrow \infty$  for every  $t > 0$ .

• **STEP 2.** Let  $\underline{u} = u^*(t + \tau, x)$ , where  $u^*$  is the Barenblatt solution with mass  $M > 0$ . We refer to [18] for a complete discussion about Barenblatt solutions. By choosing the mass  $M := \int_{\mathbb{R}^d} \underline{u} \, dx$  very small, we can find  $\tau = \tau(\varepsilon) > 0$  so that  $\underline{u}(t, x) \leq c/4$  for  $|x| \geq \delta$  and  $0 < t < t_1$ , and  $\underline{u} \leq \varepsilon/2$  for  $|x| \geq 1$ .

• **STEP 3.** We compare both continuous solutions in the exterior domain  $\Omega = \{x : |x| \geq 1\}$ . At the first time where  $\underline{u}$  touches  $u_\varepsilon$  from below at a point  $|x| > 1$ , we have  $\partial_t(u_\varepsilon - \underline{u}) \leq 0$ . Let now  $w = u_\varepsilon^m - \underline{u}^m$  to get

$$\begin{aligned} (-\Delta)^s w(x) &= k_{s,d} \int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x - y|^{d+2s}} \, dy \\ &= k_{s,d} \int_{\{|x| \leq 1\}} \frac{w(x) - w(y)}{|x - y|^{d+2s}} \, dy + k_{s,d} \int_{\{|x| \geq 1\}} \frac{w(x) - w(y)}{|x - y|^{d+2s}} \, dy = I_1 + I_2 \end{aligned}$$

We want to prove now that both  $I_1 < 0$  and  $I_2 < 0$ , which leads to a contradiction. In this way we conclude that  $\underline{u} < u_\varepsilon$  for all  $0 < t < t_1$  and all  $|x| \geq 1$ .

The fact that  $I_2 \leq 0$  comes from the fact that  $w(x) = 0$  by our choice of  $x$  and  $w(y) \geq 0$  since  $(x, t)$  is the first contact point. Due to the fact that  $w(y) > 0$  near  $|x| = 1$  and  $w$  is continuous we get  $I_2 < 0$ .

As for  $I_1$ , the denominator is like a constant in that domain and we have to estimate  $w(y)$ . We know that for  $\delta < |x| < 1$  we have  $u_\varepsilon \geq c$  and  $\underline{u} \leq c/4$ , hence  $w(y) \geq C c^m > 0$  and this contributes to the integral something that is like  $-C c^m$ , which is not small. In the small ball  $|x| \leq \delta$  we use the worst case estimate  $-w(y) \leq \underline{u}$  and  $\underline{u}(t_1, y)$  has mass at most  $M$  which is small, this contributes at most a bad term of order

$$C \int_{|x| \leq \delta} \underline{u}^m dx \leq C M^m \delta^{d(1-m)},$$

which is small if  $\delta$  and  $M$  are small (here we use  $m < 1$ ). Therefore  $I_1 < 0$ .

Moreover, one has to ensure that  $\underline{u}(0, x) < u_\varepsilon(0, x)$  for  $|x| \geq 1$ . Since  $u_\varepsilon \geq \varepsilon$  and

$$\underline{u}(0, x) = \tau^{-\alpha} F(|x| \tau^{-\beta}) \leq \tau^{-\alpha} F(\tau^{-\beta}) = c \tau^{-\alpha+\beta(d+2s)} = c \tau^{2s\beta} \leq \varepsilon$$

at least for sufficiently small  $\tau$ , recall Step 2. By the parabolic comparison theorem we conclude that  $\underline{u} < u_\varepsilon$  for all  $0 < t < t_1$  and all  $|x| \geq 1$ .

• STEP 4. We finally let  $\varepsilon \rightarrow 0$  and also  $\tau$  may go to zero, and we obtain that  $u^*(x, t) := \lim_{\varepsilon \rightarrow 0} u_\varepsilon = u$ , therefore we can conclude that  $u(x, t) \geq c/|x|^{d+2s}$  when  $|x| \gg 1$  and  $t = t_1$ .

• STEP 5. Once we have obtained the spatial lower bound at times  $t \leq t_1$ , then we can compare with a Barenblatt solution and continue the lower bound for all times, to finally get that the spatial tail of the solution  $u$  can be bounded from below by  $u \geq c/|x|^{d+2s}$  when  $|x| \gg 1$ .  $\square$

## 5 Very fast diffusion range

In the very fast diffusion range  $0 < m < m_c$ , the weighted  $L^1$  estimates of Theorem 2.2 continue to hold, but this does not allow to obtain quantitative lower bounds since technique used in the good fast diffusion range does not work anymore. One problem is that the smoothing effect does not hold for general  $L^1$  initial data, therefore the optimization of Lemma 8.6 is no more valid, since  $2s < d(1-m)$  in this range. Hence the need for new weighted  $L^1$  estimates, in the form given in Step 3 of the proof of Theorem 5.1 below. Another problem typical of this range of exponent is the presence of the extinction time, which enters directly in the estimates of Theorem 5.1. We present here a technique that is based on the careful use of weight factors.

**Theorem 5.1 (Local lower bounds I)** *Let  $u$  be a weak solution to the equation (1.1), corresponding to  $u_0 \in L^1(\mathbb{R}^d) \cap L^{p_c}(\mathbb{R}^d)$  with  $0 < m < m_c = d/(d-2s)$ ,  $0 < s < 1$  and let  $p_c = d(1-m)/(2s)$ . Let also  $T = T(u_0)$  be the finite extinction time for  $u$ . Then for every  $R_0 > 0$ , there exists a time*

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{L^1(B_{R_0})}^{1-m} \leq T(u_0), \quad (5.1)$$

such that

$$\inf_{x \in B_{R_0/2}} u(t, x) \geq K \frac{\|u_0\|_{L^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} \quad \text{if } 0 \leq t \leq t_*, \quad (5.2)$$



where  $C_*$  and  $K$  are explicit positive universal constants, that depend only on  $m, s, d$ .

The expression of the constants is

$$C_* := \frac{k_{s,d} \omega_d^m}{4^{d+1-2s}}, \quad K := \left( \frac{k_{s,d}}{4^{3d+1-2s} d} \right)^{\frac{1}{m}}, \quad (5.3)$$

where  $k_{s,d}$  is the constant of the representation formula  $\varphi(x) = k_{s,d} \int_{\mathbb{R}^d} \frac{\rho(y)}{|x-y|^{d-2s}} dy$  and  $\omega_d$  is the volume of the unit ball.

*Proof. of Theorem 5.1* It is divided into several steps as follows.

• **STEP 1. Reduction.** By the comparison principle that it is sufficient to prove lower bounds for solutions  $u$  to the following reduced problem:

$$\begin{cases} \partial_t u + (-\Delta)^s(u^m) = 0, & \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 \chi_{B_{R_0}} = \underline{u}_0, & \text{in } \mathbb{R}^d, \end{cases} \quad (5.4)$$

where  $m > 1$ ,  $0 < s < 1$ , and  $R_0 > 0$ . We only assume that  $0 \leq u_0 \in L^1(B_{R_0})$ , which implies that  $\underline{u}_0 \in L^1(\mathbb{R}^d)$  since  $\text{supp}(u_0) \subseteq B_{R_0}$  and also that  $\|u_0\|_{L^1(\mathbb{R}^d)} = \|u_0\|_{L^1(B_{R_0})}$ . It is not restrictive to assume that the ball  $B_{R_0}$  is centered at the origin. We call  $M_0 = \|u_0\|_{L^1(B_{R_0})}$ .

• **STEP 2. Aleksandrov principle.** We recall Theorem 11.2 of [18]. In view of the fact that the initial function is supported in the ball  $B_{R_0}(0)$ , we have that

$$u(t, 0) \geq u(t, x), \quad \text{for all } t > 0 \text{ and } |x| \geq 2R_0.$$

Therefore, one has

$$\sup_{x \in \mathbb{R}^d \setminus B_{2R_0}} u(t, x) \leq u(t, 0). \quad (5.5)$$

• **STEP 3.  $L^1$  Weighted estimates.** Choose a test function  $\varphi \geq 0$  such that  $-(-\Delta)^s \varphi = \rho$  with  $\rho = 0$  on  $B_{2R_0}$  and on  $B_{R_1}^c$ , and  $0 < \rho \leq 1$  on the annulus  $A := B_{R_1} \setminus B_{2R_0}$ , with  $0 < 2R_0 \leq R_1$ , and  $R_0$  as in Step 1, such that  $\text{supp}(u_0) \subseteq B_{R_0}$ . Using the explicit representation of  $\varphi$  in terms  $\rho$  and the integral kernel  $K(x, y) = k_{s,d} |x - y|^{n-2s}$  we get the estimates

$$\varphi(x) = k_{s,d} \int_{\mathbb{R}^d} \frac{\rho(y)}{|x-y|^{d-2s}} dy \geq \frac{k_{s,d} \|\rho\|_1}{(R_1 + R_0)^{d-2s}} \geq k_0 > 0, \quad \text{for all } x \in B_{R_0}(0),$$

since  $|x - y| \leq R_0 + R_1$ . We can always choose  $\rho \geq 1/2$  on the smaller annulus  $A_0 = B_{2R_0+3(R_1-2R_0)/4} \setminus B_{2R_0+(R_1-2R_0)/4} \subseteq A$ , so that

$$\begin{aligned} \|\rho\|_1 &= \int_{A_1} \rho(x) dx \geq \int_{A_0} \rho(x) dx \geq \frac{|A_0|}{2} = \frac{1}{2} |B_{2R_0+3(R_1-2R_0)/4} \setminus B_{2R_0+(R_1-2R_0)/4}| \\ &= \frac{\omega_d}{2} \left[ \left( 2R_0 + \frac{1}{4}(R_1 - 2R_0) + (R_1 - 2R_0) \right)^d - \left( 2R_0 + \frac{1}{4}(R_1 - 2R_0) \right)^d \right] \geq \frac{\omega_d}{2} (R_1 - 2R_0)^d \end{aligned}$$

since  $(a + b)^d - a^d \geq b^d$  for any  $a, b \geq 0$ . Then  $k_0 > 0$  takes the form

$$k_0 := \frac{k_{s,d} \omega_d}{2} \frac{(R_1 - 2R_0)^d}{(R_1 + R_0)^{d-2s}} \quad (5.6)$$

Now we observe that letting  $T = T(u_0) > 0$  be the finite extinction time for the reduced problem (5.4), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} u_0(x) \varphi \, dx &= \int_{\mathbb{R}^d} u(0, x) \varphi \, dx - \int_{\mathbb{R}^d} u(T, x) \varphi \, dx = - \int_0^T \int_{\mathbb{R}^d} \partial_\tau u(\tau, x) \varphi(x) \, dx d\tau \\
&= \int_0^T \int_{\mathbb{R}^d} (-\Delta)^s (u^m(\tau, x)) \varphi(x) \, dx d\tau = \int_0^T \int_{\mathbb{R}^d} u^m(\tau, x) (-\Delta)^s \varphi(x) \, dx d\tau \\
&= \int_0^T \int_A u^m(\tau, x) \rho(x) \, dx d\tau \\
&= \int_0^{t_*} \int_A u^m(\tau, x) \rho(x) \, dx d\tau + \int_{t_*}^T \int_A u^m(\tau, x) \rho(x) \, dx d\tau := (I) + (II)
\end{aligned} \tag{5.7}$$

where  $0 \leq t_* \leq T$  will be chosen later.

Next we estimate (I). We first observe that

$$\begin{aligned}
\int_A u^m(\tau, x) \rho(x) \, dx &\leq \int_{B_{R_1}} u^m(\tau, x) \, dx \leq |B_{R_1}|^{1-m} \left( \int_{B_{R_1}} u(\tau, x) \, dx \right)^m \\
&\leq |B_{R_1}|^{1-m} \left( \int_{\mathbb{R}^d} u(\tau, x) \, dx \right)^m \leq |B_{R_1}|^{1-m} \left( \int_{\mathbb{R}^d} \underline{u}_0(x) \, dx \right)^m \\
&\leq |B_{R_1}|^{1-m} M_0^m
\end{aligned}$$

since  $0 < m < 1$ ,  $0 < \rho \leq 1$ , and in the last step we have used the fact that  $\|u(t)\|_{L^1(\mathbb{R}^d)} \leq \|\underline{u}_0\|_{L^1(\mathbb{R}^d)}$ , which has been proven in [12], together with the fact that  $M_0 = \|u_0\|_{L^1(B_{R_0})} = \|\underline{u}_0\|_{L^1(\mathbb{R}^d)}$ . Therefore,

$$(I) := \int_0^{t_*} \int_A u^m(\tau, x) \rho(x) \, dx d\tau \leq |B_{R_1}|^{1-m} t_* M_0^m. \tag{5.8}$$

We now estimate (II) by using the Aleksandrov principle:

$$\begin{aligned}
(II) := \int_{t_*}^T \int_A u^m(\tau, x) \rho(x) \, dx d\tau &\leq \int_{t_*}^T \int_A u^m(\tau, x) \, dx d\tau \stackrel{(a)}{=} (T - t_*) \int_A u^m(\tau_1, x) \, dx \\
&\leq (T - t_*) |A| \sup_{x \in A} u^m(\tau_1, x) \leq (T - t_*) |A| u^m(\tau_1, 0)
\end{aligned} \tag{5.9}$$

where in (a) we have used the mean value theorem for the function  $U(\tau) = \int_A u^m(\tau, x) \, dx$  so that there exists a  $\tau_1 \in [t_*, T]$  such that  $\int_{t_*}^T U(\tau) \, d\tau = (T - t_*) U(\tau_1)$ . In (b) we have used the Aleksandrov principle, which gives  $\sup_{x \in A} u^m(\tau_1, x) \leq u^m(\tau_1, 0)$ . Summing up, we have obtained, joining (5.7), (5.8) and (5.9)

$$\int_{\mathbb{R}^d} u_0(x) \varphi \, dx \leq |B_{R_1}|^{1-m} t_* M_0^m + (T - t_*) |A| u^m(\tau_1, 0) \tag{5.10}$$

for some  $\tau_1 \in [t_*, T]$ . In addition, we have  $\int_{\mathbb{R}^d} u_0(x) \varphi \, dx \geq M_0 k_0$ . We finally remark that from inequality (5.10) we get a lower bound for the extinction time, just by letting  $t_* = T$  in formula (5.10):

$$k_0 M_0 \leq \int_{\mathbb{R}^d} u_0(x) \varphi \, dx \leq |B_{R_1}|^{1-m} T M_0^m, \quad \text{that is} \quad T \geq k_0 \frac{M_0^{1-m}}{|B_{R_1}|^{1-m}} \tag{5.11}$$

• STEP 4. *Choosing the critical time  $t_*$ .* We now choose  $t_*$  to be small enough, more precisely

$$t_* := \frac{k_0}{2} \frac{M_0^{1-m}}{|B_{R_1}|^{1-m}} \leq T, \quad (5.12)$$

we note that  $t_* \leq T$  follows by (5.11). With this choice of  $t_*$ , inequality (5.10) becomes

$$\frac{k_0}{2} M_0 = k_0 M_0 - |B_{R_1}|^{1-m} t_* M_0^m \leq (T - t_*) |A| u^m(\tau_1, 0) \leq T |A| u^m(\tau_1, 0) \quad (5.13)$$

which is the desired positivity estimate at a time  $\tau_1 \in [t_*, T]$ , namely

$$\frac{k_0 M_0}{2T |B_{R_1} \setminus B_{2R_0}|} \leq u^m(\tau_1, 0) \quad (5.14)$$

• STEP 5. *Positivity backward in time.* Using Benilan-Crandall estimates which depend only by the homogeneity of the equations, cf. [3]

$$u_t \leq \frac{u}{(1-m)t} \quad (5.15)$$

we can prove positivity in the time interval  $[0, \tau_1]$ . These estimates in the fractional case has been proven in [12], and imply that the function:  $u(t, x)t^{-1/(1-m)}$  is non-increasing in time, thus for any  $t \in [0, \tau_1]$  we have that

$$\begin{aligned} u(t, 0) &\geq \frac{t^{\frac{1}{1-m}}}{\tau_1^{\frac{1}{1-m}}} u(\tau_1, 0) \geq \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{1-m}}} u(\tau_1, 0) \geq \left[ \frac{k_0 M_0}{2T |B_{R_1} \setminus B_{2R_0}|} \right]^{\frac{1}{m}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{1-m}}} \\ &= \left[ \frac{k_{s,d}}{4(R_1 + R_0)^{d-2s}} \frac{(R_1 - 2R_0)^d}{R_1^d - (2R_0)^d} \right]^{\frac{1}{m}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} M_0^{\frac{1}{m}} \end{aligned} \quad (5.16)$$

since  $t_* \leq \tau_1 \leq T$ . Moreover we have that

$$\begin{aligned} u(t, 0) &\geq \left[ \frac{k_{s,d}}{4(R_1 + R_0)^{d-2s}} \frac{(R_1 - 2R_0)^{d-1}}{d(2R_0)^{d-1}} \right]^{\frac{1}{m}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} M_0^{\frac{1}{m}} \\ &= \left( \frac{k_{s,d}}{4d} \right)^{1/m} \left( \frac{R_1}{2R_0} - 1 \right)^{\frac{d-1}{m}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} \frac{M_0^{\frac{1}{m}}}{(R_1 + R_0)^{\frac{d-2s}{m}}} \end{aligned} \quad (5.17)$$

where we have used the numerical inequality  $a^d - b^d \leq da^{d-1}(a - b)$ , valid for any  $a = R_1 > 2R_0 = b$  to pass from (5.16) to (5.17). By a standard argument it is easy to pass from the center to the infimum on  $B_{R_0/2}(0)$  in the above estimates. The proof is concluded once we let  $R_1 = 3R_0$ .  $\square$

**Remarks.** (i) This result can be written alternatively as saying that there exists a universal constant  $K_1 = \max\{K^{-m}, C_*^{1/(1-m)}\}$  such for all solutions in the above class we have: for any  $0 \leq t \leq T$  and  $R > 0$

$$\frac{\|u_0\|_{L^1(B_R)}}{R^d} \leq K_1 \left[ \frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{T^{\frac{1}{1-m}}}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t, x) \right]. \quad (5.18)$$

This is easy to prove: by the previous Theorem, we have that either  $t_* \leq t$ , that is

$$\frac{\|u_0\|_{L^1(B_R)}}{R^d} \leq \left[ \frac{t}{C_* R^{2s}} \right]^{\frac{1}{1-m}}$$

or that  $0 \leq t \leq t_*$  and (5.2) holds, namely

$$\frac{\|u_0\|_{L^1(B_R)}}{R^d} \leq \frac{T^{\frac{1}{1-m}}}{K^m t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t, x)$$

therefore, letting  $K_1 = \max\{K^{-m}, C_*^{1/(1-m)}\}$  we get (5.18).

This equivalent version is in complete formal agreement with similar estimates proved by the authors in [6], in the case  $s = 1$ . However, our proof below differs from the one in [6], and provides an alternative proof when  $s = 1$ . On the other hand, here we are considering solutions to the Cauchy problem, while in [6] we consider local weak solutions (i.e. without specifying boundary conditions). These estimates have been called Aronson-Caffarelli estimates in [6], when  $s = 1$ , since they are quite similar to the one that can be obtained for  $m > 1$ , see Section 6. Finally we shall remark that in Section 5.1 we will obtain quantitative upper estimates on the extinction time, and this will help to eliminate  $T$  from the above lower estimates.

(ii) By comparison it is easy to prove that this estimates hold for a larger class of solutions, more precisely for the class of very weak solutions to the Cauchy Problem (1.1) constructed in Theorem 3.1, Section 3. This implies that the positivity result holds for solutions  $u(t, \cdot) \in L^1(\mathbb{R}^d, \varphi dx)$  corresponding to initial data  $0 \leq u_0 \in L^1(\mathbb{R}^d, \varphi dx)$ , where  $\varphi$  is as in Theorem 2.2 with decay at infinity  $|x|^{-\alpha}$ ,  $d - [2s/(1-m)] < \alpha < d + (2s/m)$ .

Once comparison is used, we can use as  $T$  the extinction time of the reduced problem 5.4 in Step 1 of the above proof. In this way the quantitative result applies to solutions  $u$  that may not extinguish in finite time. Therefore we can interpret  $T$  as the *minimal life time for the solution*  $u(t, \cdot)$ , a concept that was already introduced by the authors in [6], for which formula (5.1) provides a quantitative lower bound, namely

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{L^1(B_{R_0})}^{1-m} \leq T(u_0). \quad (5.19)$$

**Corollary 5.2 (Solutions that do not extinguish in finite time)** *Let  $0 < m < m_c$  and consider an initial datum  $0 \leq u_0 \in L^1(\mathbb{R}^d, \varphi dx)$ , where  $\varphi$  is as in Theorem 2.2, in particular, when  $u_0 \in L^1(\mathbb{R}^d)$ . Assume moreover that*

$$\liminf_{R \rightarrow +\infty} R^{\frac{2s}{1-m}-d} \|u_0\|_{L^1(B_R)} = +\infty. \quad (5.20)$$

*Then the corresponding solution  $u(t, x)$  exists and is positive globally in space and time, hence does not extinguish in finite time. Moreover the quantitative lower bounds (5.2) of Theorem 5.1 hold for any  $0 \leq t \leq t_*$  with  $t_*$  given in (5.1) and  $T = T(u_0 \chi_{B_{R_0}}) < +\infty$  is the extinction time of a reduced problem.*

*Proof.* If we consider an initial data with that behaviour at infinity, then by Theorem 3.1 there exists a very weak solution. By letting  $R \rightarrow +\infty$  in the above lower bound (5.19) for  $T$ , to conclude that the minimal life time  $T(u_0 \chi_{B_R}) \rightarrow \infty$ , recalling that in this very fast diffusion range we have  $2s < d(1-m)$ , since  $0 < m < m_c$ .  $\square$

**Remark.** A practical assumption on the initial datum  $u_0$  that implies (5.20) is

$$\liminf_{|x| \rightarrow +\infty} |x|^{\frac{2s}{1-m}} u_0(x) = +\infty. \quad (5.21)$$

In view of Proposition 5.3 below, the exponent is sharp.

### 5.1 Estimating the extinction time.

We next estimate the extinction time in terms of the initial data, extending a classical result of Benilan and Crandall [2]. This is needed to eliminate the dependence on  $T$  in the above lower estimates when we consider initial data in  $L^1(\mathbb{R}^d) \cap L^{p_c}(\mathbb{R}^d)$ . For a detailed study of extinction time in the standard fast diffusion equation, see [16].

**Proposition 5.3 (Upper bounds for the extinction time)** *Let  $u$  be a weak solution to the equation (1.1), corresponding to  $u_0 \in L^1(\mathbb{R}^d) \cap L^{p_c}(\mathbb{R}^d)$  with  $0 < m < m_c = d/(d-2s)$ ,  $0 < s < 1$  and let  $p_c = d(1-m)/(2s)$ . Then for all  $0 \leq s \leq t$  the following estimate holds true*

$$\left[ \int_{\mathbb{R}^d} |u(t, x)|^{p_c} dx \right]^{\frac{2s}{d}} \leq \left[ \int_{\mathbb{R}^d} |u(s, x)|^{p_c} dx \right]^{\frac{2s}{d}} - \frac{4m[d(1-m) - 2s]}{d(d-2s)\mathcal{S}_s^2} (t-s) \quad (5.22)$$

Moreover, there exists a finite extinction time  $T \geq 0$  which can be bounded above as follows

$$T \leq \frac{d(d-2s)\mathcal{S}_s^2}{4m[d(1-m) - 2s]} \|u_0\|_{L^{p_c}(\mathbb{R}^d)}^{1-m}. \quad (5.23)$$

*Proof.* The proof presented below is analogous to the one of Theorem 9.5 of [12], but here we pay attention to the quantitative estimates. We multiply the equation by  $|u|^{p-2}u$  with  $p > 1$ , and integrate in  $\mathbb{R}^d$ . Using Strook-Varopoulos inequality (8.6) in the form (8.7), we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |u(t, x)|^p dx &= -p \int_{\mathbb{R}^d} |u|^{p-2} u (-\Delta)^s (|u|^{m-1} u) dx \\ &\leq -\frac{4mp(p-1)}{(p+m-1)^2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} |u|^{\frac{p+m-1}{2}} \right|^2 dx \\ &\leq -\frac{4mp(p-1)}{(p+m-1)^2 \mathcal{S}_s^2} \left( \int_{\mathbb{R}^d} |u|^{\frac{d(p+m-1)}{d-2s}} dx \right)^{\frac{d-2s}{d}} \end{aligned} \quad (5.24)$$

where in the last step we have used the Sobolev inequality (8.8) applied to  $f = |u|^{\frac{p+m-1}{2}}$ . Now we make the choice  $p = p_c = d(1-m)/2s$ , so that  $p_c = \frac{d(p_c+m-1)}{d-2s}$ , and we know that  $p_c > 1$  if and only if  $m < m_c = d/(d-2s)$ , and inequality (5.24) becomes

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(t, x)|^{p_c} dx \leq -\frac{4m[d(1-m) - 2s]}{2s(d-2s)\mathcal{S}_s^2} \left( \int_{\mathbb{R}^d} |u(t, x)|^{p_c} dx \right)^{1-\frac{2s}{d}} \quad (5.25)$$

Integrating the above differential inequality on  $(s, t)$  gives both (5.22) and inequality (5.23).  $\square$

Thanks to the above estimates we can get rid of the extinction time  $T$  in the lower estimates of Theorem 5.1.

**Theorem 5.4 (Local lower bounds II)** *Let  $u$  be a weak solution to the equation (1.1), corresponding to  $u_0 \in L^1(\mathbb{R}^d) \cap L^{p_c}(\mathbb{R}^d)$  with  $0 < m < m_c = d/(d-2s)$ ,  $0 < s < 1$  and let  $p_c = d(1-m)/(2s)$ . Then for every ball  $B_{2R_0} \subset \Omega$ , there exists a time*

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{L^1(B_{R_0})}^{1-m} \leq T(u_0) \leq \overline{C} \|u_0\|_{L^{p_c}(\mathbb{R}^d)}^{1-m}, \quad (5.26)$$

where we recall that  $T(u_0)$  is the finite extinction time, such that

$$\inf_{x \in B_{R_0/2}} u(t, x) \geq K_2 \frac{\|u_0\|_{L^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{\|u_0\|_{L^{p_c}(B_{R_0})}^{\frac{d}{2m}}} \quad \text{if } 0 \leq t \leq t_*, \quad (5.27)$$

where  $C_*$  and  $K_1$  are explicit positive universal constants, that depend only on  $m, s, d$ .

The expression of the constants

$$K_2 := K \left[ \frac{4m[d(1-m) - 2s]}{d(d-2s)\mathcal{S}_s^2} \right]^{\frac{1}{m(1-m)}}, \quad \overline{C} := \frac{d(d-2s)\mathcal{S}_s^2}{4m[d(1-m) - 2s]} \quad (5.28)$$

where  $C_*$  and  $K$  are as in (5.2) and  $k_{s,d}$  is the constant of the representation formula  $\varphi(x) = k_{s,d} \int_{\mathbb{R}^d} \frac{\rho(y)}{|x-y|^{d-2s}} dy$ .

**Remark.** This result can be written alternatively as saying that there exists a universal constant  $K_3 = \max\{K_2^{-m}, C_*^{1/(1-m)}\}$  such for all solutions in the above class we have: for any  $0 \leq t \leq T$  and  $R > 0$

$$\frac{\|u_0\|_{L^1(B_R)}}{R^d [1 \vee \|u_0\|_{L^{p_c}(B_R)}]} \leq K_3 \left[ \frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{1}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t, x) \right] \quad (5.29)$$

This equivalent version is in complete formal agreement with similar estimates proved by the authors in [6], in the case  $s = 1$ .

*Proof of Theorem 5.4.* The proof is the same as for Theorem 5.1, as far as the first 3 steps are concerned. At the end of Step 3, we need to bound from above the extinction time for the reduced problem (5.4) with the estimates (5.23) which give

$$T \leq \frac{d(d-2s)\mathcal{S}_s^2}{4m[d(1-m) - 2s]} \left[ \int_{\mathbb{R}^d} |u_0(x)|^{p_c} dx \right]^{\frac{2s}{d}} = \frac{d(d-2s)\mathcal{S}_s^2}{4m[d(1-m) - 2s]} \left[ \int_{B_{R_0}} |u_0(x)|^{p_c} dx \right]^{\frac{2s}{d}}, \quad (5.30)$$

since  $\text{supp}(u_0) \subseteq B_{R_0}$ . Then the proof follows simply by replacing  $T$  with the above upper bound.  $\square$

## 6 The Porous medium case

Lower estimates for nonnegative solutions of the standard porous medium equation were obtained in Aronson-Caffarelli in a famous paper [1]. We want to show in this section how such a priori estimates extend to the fractional version considered in this paper.

**Theorem 6.1 (Local lower bound)** *Let  $u$  be a weak solution to Equation (1.1), corresponding to  $u_0 \in L^1(\mathbb{R}^d)$ . and let  $m > 1$ . We put  $\vartheta := 1/[2s + d(m-1)] > 0$ . Then there exists a time*

$$t_* := C R^{2s+d(m-1)} \|u_0\|_{L^1(B_R)}^{-(m-1)} \quad (6.1)$$

*such that for every  $t \geq t_*$  we have the lower bound*

$$\inf_{x \in B_{R/2}} u(t, x) \geq K \frac{\|u_0\|_{L^1(B_R)}^{2s\vartheta}}{t^{d\vartheta}} \quad (6.2)$$

*valid for all  $R > 0$ . The positive constants  $C$  and  $K$  depend only on  $m, s$  and  $d$ , and not on  $R$ .*

**Remark.** This result can be written alternatively as saying that there exists a universal constant  $C_1 = C_1(d, s, m)$  such for all solutions in the above class we have

$$\int_{B_R(0)} u_0(x) dx \leq C_1 \left( R^{1/\vartheta(m-1)} t^{-1/(m-1)} + u(0, t)^{1/2s\vartheta} t^{d/2s} \right). \quad (6.3)$$

This equivalent version is in complete formal agreement with Aronson-Caffarelli's estimate for  $s = 1$ . However, our proof below differs very strongly from the ideas used in Aronson-Caffarelli's case since we cannot use the property of finite propagation of solution with compact support, which is false for  $s < 1$ .

*Proof.* It is divided into several steps as follows.

• **STEP 1. Reduction.** By the comparison principle that it is sufficient to prove lower bounds for solutions  $u$  to the following reduced problem:

$$\begin{cases} \partial_t u + (-\Delta)^s(u^m) = 0, & \text{in } (0, \infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 \chi_{B_{R_0}} = \underline{u}_0, & \text{in } \mathbb{R}^d, \end{cases} \quad (6.4)$$

where  $m > 1$ ,  $0 < s < 1$ , and  $R_0 > 0$ . We only assume that  $0 \leq u_0 \in L^1(B_{R_0})$ , which implies that  $\underline{u}_0 \in L^1(\mathbb{R}^d)$  since  $\text{supp}(\underline{u}_0) \subseteq B_{R_0}$  and also that  $\|\underline{u}_0\|_{L^1(\mathbb{R}^d)} = \|u_0\|_{L^1(B_{R_0})}$ . It is not restrictive to assume that the ball  $B_{R_0}$  is centered at the origin.

• **STEP 2. Smoothing effects.** In [12] there are the global  $L^1$ - $L^\infty$  smoothing effects, which can be applied to solutions to our reduced Problem 6.4 as follows:

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{I_\infty}{t^{d\vartheta}} \|\underline{u}_0\|_{L^1(\mathbb{R}^d)}^{2s\vartheta} = \frac{I_\infty}{t^{d\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} \quad (6.5)$$

where  $\vartheta = 1/[2s + d(m-1)]$  and the constant  $I_\infty$  only depends on  $d, s, m$ .

• **STEP 3. Aleksandrov principle.** We recall Theorem 11.2 of [18]. In view of the fact that the initial function is supported in the ball  $B_{R_0}(0)$ , we have that

$$u(t, 0) \geq u(t, x), \quad \text{for all } t > 0 \text{ and } |x| \geq 2R_0.$$

Therefore, one has

$$\sup_{x \in \mathbb{R}^d \setminus B_{2R_0}} u(t, x) \leq u(t, 0). \quad (6.6)$$

• **STEP 4. Weighted estimates.** If  $\psi$  is a smooth, nonnegative, and sufficiently decaying function, we have

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) \psi(x) dx \right| &= \left| \int_{\mathbb{R}^d} ((-\Delta)^s u^m) \psi dx \right| =_{(a)} \left| \int_{\mathbb{R}^d} u^m (-\Delta)^s \psi dx \right| \\ &\leq \|u(t)\|_{L^\infty(\mathbb{R}^d)}^{m-1} \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} u(t, x) dx \\ &\leq_{(b)} \frac{I_\infty^{m-1}}{t^{d\vartheta(m-1)}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta(m-1)} \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \underline{u}_0(x) dx \\ &:= \frac{I_\infty^{m-1}}{t^{d\vartheta(m-1)}} \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^d)} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta(m-1)+1} := \frac{K[u_0, \psi]}{t^{d\vartheta(m-1)}}. \end{aligned}$$

Notice that in (a) we have used the fact that  $(-\Delta)^s$  is a symmetric operator, while in (b) we have used the smoothing effect (6.5) of Step 2 and the conservation of mass:  $\int_{\mathbb{R}^d} u(t, x) dx = \int_{\mathbb{R}^d} u_0(x) dx$ , for all  $t > 0$ , together with the fact that  $\text{supp}(u_0) \subseteq B_{R_0}$ . We refer to [12] for a proof of the smoothing effect and of the conservation of mass. Summing up,

$$\left| \frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) \psi(x) dx \right| \leq \frac{K[u_0, \psi]}{t^{1-2s\vartheta}},$$

since  $d\vartheta(m-1) = 1 - 2s\vartheta$ . Integrating the above differential inequality on  $(0, t)$  with  $t \geq 0$  we obtain:

$$-\frac{K[u_0, \psi]}{2s\vartheta} t^{2s\vartheta} \leq \int_{\mathbb{R}^d} u(t, x) \psi(x) dx - \int_{\mathbb{R}^d} u(0, x) \psi(x) dx \leq \frac{K[u_0, \psi]}{2s\vartheta} t^{2s\vartheta}.$$

We will use this in the form

$$\int_{\mathbb{R}^d} u(0, x) \psi(x) dx - \frac{K[u_0, \psi]}{2s\vartheta} t^{2s\vartheta} \leq \int_{\mathbb{R}^d} u(t, x) \psi(x) dx. \quad (6.7)$$

Moreover, if  $\psi \in L^1(\mathbb{R}^d)$  and  $R_1 \geq 2R_0$ , we have

$$\begin{aligned} \int_{\mathbb{R}^d} u(t, x) \psi(x) dx &= \int_{B_{R_1}} u(t, x) \psi(x) dx + \int_{B_{R_1}^c} u(t, x) \psi(x) dx \\ &\leq_{(a)} |B_{R_1}| \sup_{|x| \leq R_1} u(t, x) + \sup_{x \in \mathbb{R}^d \setminus B_{2R_0}} u(t, x) \int_{B_R^c} \psi(x) dx \\ &\leq_{(b)} |B_{R_1}| \frac{I_\infty}{t^{d\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} + u(t, 0) \int_{B_{R_1}^c} \psi(x) dx \end{aligned} \quad (6.8)$$

where in (a) we have used the fact that  $\psi \leq 1$ ,  $R_1 \geq 2R_0$ , and that  $\psi \in L^1(\mathbb{R}^d)$ . In (b) we have used the smoothing effect (6.5) of step 2 and the Aleksandrov principle of Step 3. Putting together inequalities (6.7) and (6.8), we obtain

$$\int_{\mathbb{R}^d} u(0, x) \psi(x) dx - \frac{K[u_0, \psi]}{2s\vartheta} t^{2s\vartheta} - |B_{R_1}| \frac{I_\infty}{t^{d\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} \leq u(t, 0) \int_{B_{R_1}^c} \psi(x) dx. \quad (6.9)$$

Next, in order to estimate  $K[u_0, \psi]$  in a convenient way we take  $\psi(x) = \phi(|x|/R)$  with  $\phi$  as in Lemma 2.1, we have  $|(-\Delta)^s \psi| \leq c_3 R^{-2s}$ , for some constant  $c_3 = c_3(d, s)$ . Then,

$$K[u_0, \psi] = I_\infty^{m-1} \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^d)} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta(m-1)+1} \leq \frac{c_3 I_\infty^{m-1}}{R^{2s}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta(m-1)+1} \quad (6.10)$$

When  $R \geq R_0$  and  $R_1 \geq 2R_0$ , we arrive at

$$\begin{aligned} \|u_0\|_{L^1(B_{R_0})} - \frac{c_3 I_\infty^{m-1}}{2s\vartheta} \frac{\|u_0\|_{L^1(B_{R_0})}^{2s\vartheta(m-1)+1}}{R^{2s}} t^{2s\vartheta} - \omega_d R_1^d \frac{I_\infty}{t^{d\vartheta}} \|u_0\|_{L^1(B_{R_0})}^{2s\vartheta} &\leq u(t, 0) \int_{B_{R_1}^c} \psi(x) dx \\ &\leq u(t, 0) R^d \int_{\mathbb{R}^d} \varphi(x) dx = c_4 R^d u(t, 0). \end{aligned} \quad (6.11)$$



• **STEP 5. Choosing the parameters.** We want to choose  $t > 0$ ,  $R \geq R_0$  and  $R \geq 2R_0$  so that the left-hand side of (6.11) is larger than  $\|u_0\|_{L^1(B_{R_0})}/2$ , which will then give the desired bound from below for  $u(0, t)$ . We first make the choice

$$R_1^d = \frac{\|u_0\|_{L^1(B_{R_0})}^{d(m-1)\vartheta}}{4\omega_d I_\infty} t^{d\vartheta}, \quad (6.12)$$

which will satisfy the condition  $R_1 \geq 2R_0$  if and only if  $t \geq t_*$  where

$$t_*^\vartheta = c_5 R_0 \|u_0\|_{L^1(B_{R_0})}^{-(m-1)\vartheta}, \quad c_5 = 2^{1+(2/d)(\omega_d I_\infty)^{1/d}}. \quad (6.13)$$

Now we can make the second choice,  $R$  has to be large enough, for instance:

$$R = c_6 \|u_0\|_{L^1(B_{R_0})}^{\vartheta(m-1)} t^\vartheta, \quad c_6 = \left( \frac{4 c_3 I_\infty^{m-1}}{2s\vartheta} \right)^{1/2s} \quad (6.14)$$

Both choices will give for  $t \geq t_*$  the lower bound

$$\frac{\|u_0\|_{L^1(B_{R_0})}}{2c_4 R^d} \leq u(t, 0),$$

which can be rewritten as

$$c_7 \frac{\|u_0\|_{L^1(B_{R_0})}^{2s\vartheta}}{t^{d\vartheta}} \leq u(t, 0), \quad \text{for any } t \geq t_*, \quad \text{with } c_7 = \frac{1}{2c_4 c_6^d}. \quad (6.15)$$

By a standard argument it is easy to pass from the center to the infimum on  $B_{R_0/2}(0)$  in the above estimates.  $\square$

**Remark.** In the limit  $m \rightarrow 1$  of the estimate of Theorem 6.1 we obtain the result of Proposition 4.2 for  $m = 1$ .

**Open Problem.** To calculate the positivity of the solutions for small times is not known yet.

## 7 Existence and uniqueness of initial traces

The existence of solutions of the Cauchy Problem (1.1)-(1.2) can be extended to the case where the initial datum is a finite and nonnegative Radon measure. We denote by  $\mathcal{M}^+(\mathbb{R}^d)$  the space of such measures on  $\mathbb{R}^d$ . Here is the result proved in Theorem 4.1 of [18].

**Theorem.** For every  $\mu \in \mathcal{M}^+(\mathbb{R}^d)$  there exists a nonnegative and continuous weak solution of Equation (1.1) in  $Q = (0, \infty) \times \mathbb{R}^d$  taking initial data  $\mu$  in the sense that for every  $\varphi \in C_c^2(\mathbb{R}^d)$  we have

$$\lim_{t \rightarrow 0^+} \int u(t, x) \varphi(x) dx = \int \varphi(x) d\mu(x). \quad (7.1)$$

In this section we address the reverse problem, i.e., given a solution to find the initial trace. In the case  $s = 1$  such question was solved thanks to the works of Aronson-Caffarelli [1], Dahlberg-Kenig [8], Pierre [13] and others, see a presentation in [17], Chapter 13.

**Lemma 7.1 (Conditions for existence and uniqueness of initial traces)** *Let  $m > 0$  and let  $u$  be a solution to equation (1.1) in  $(0, T] \times \mathbb{R}^d$ . Assume that there exist a time  $0 < T_1 \leq T$ , some positive constants  $K_1, K_2, \alpha > 0$  and a continuous function  $\omega : [0, +\infty) \rightarrow [0, +\infty)$ , with  $\omega(0) = 0$  such that*

$$(i) \quad \sup_{t \in (0, T_1]} \int_{B_R(x_0)} u(t, x) \, dx \leq K_1, \quad \forall R > 0, x_0 \in \mathbb{R}^d, \quad (7.2)$$

as well as

$$(ii) \quad \left[ \int_{\mathbb{R}^d} u(t, x) \varphi(x) \, dx \right]^\alpha \leq \left[ \int_{\mathbb{R}^d} u(t', x) \varphi(x) \, dx \right]^\alpha + K_2 \omega(|t - t'|) \quad (7.3)$$

for all  $0 < t, t' \leq T_1$  and for all  $\varphi \in C_c^\infty(\mathbb{R}^d)$ . Then there exists a unique nonnegative Radon measure  $\mu$  as initial trace, that is

$$\int_{\mathbb{R}^d} \varphi \, d\mu = \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^d} u(t, x) \varphi(x) \, dx, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d).$$

Moreover the initial trace  $\mu$  satisfies the bound (7.2) with the same constant, namely  $\mu(B_R(x_0)) \leq K_1$ .

Notice that the constants  $K_1$  and  $K_2$  may depend on  $u$  and  $\varphi$ , usually through some norm.

*Proof.* The proof is divided in two steps in which we prove existence and uniqueness of the initial trace respectively.

• **STEP 1. Existence of the initial trace.** Hypothesis (i) easily implies that

$$\limsup_{t \rightarrow 0^+} \int_{B_R(x_0)} u(t, x) \, dx \leq K_1, \quad \forall R > 0, x_0 \in \mathbb{R}^d.$$

Moreover, it implies weak compactness for measures (to be more precise, weak\* compactness, see Theorem 8.8 in the Appendix 8.6), so that there exists a sequence  $t_k \rightarrow 0^+$  as  $k \rightarrow \infty$  with  $0 < t_k < T_1$ , and a nonnegative Radon measure  $\mu$  so that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} u(t_k, x) \varphi(x) \, dx = \int_{\mathbb{R}^d} \varphi \, d\mu \quad \text{for all } \varphi \in C_c^0(\mathbb{R}^d).$$

The bound (7.9) on the initial trace:  $\mu(B_R(x_0)) \leq K_1$  follows from the above bound on the lim sup.

• **STEP 2. Uniqueness of the initial trace.** The initial trace whose existence we have just proved may, of course, depend on the sequence  $t_k$ . We will now show that this is not the case, thanks to hypothesis (ii). Assume that there exist two sequences  $t_k \rightarrow 0^+$  and  $t'_k \rightarrow 0^+$  as  $k \rightarrow \infty$ , so that  $u(t_k) \rightarrow \mu$  and  $u(t'_k) \rightarrow \nu$ , with  $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^d)$ . We will prove that

$$\int_{\mathbb{R}^d} \varphi \, d\mu = \int_{\mathbb{R}^d} \varphi \, d\nu \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d). \quad (7.4)$$

so that  $\mu = \nu$  as positive linear functionals on  $C_c^\infty(\mathbb{R}^d)$ . Then by the Riesz representation theorem (cf. Theorem 8.9) we know that  $\mu = \nu$  also as Radon measures on  $\mathbb{R}^d$ . Therefore, it only remains to prove (7.4): hypothesis (ii) implies that for any  $t, t' > 0$ , with  $0 < t + t' \leq T_1 \leq T$ , and any  $\varphi \in C_c^\infty(\mathbb{R}^d)$  we have  $\omega(|(t + t') - t|) = \omega(t')$  and

$$\left[ \int_{\mathbb{R}^d} u(t, x) \varphi(x) \, dx \right]^\alpha \leq \left[ \int_{\mathbb{R}^d} u(t + t', x) \varphi(x) \, dx \right]^\alpha + K_2 \omega(t'). \quad (7.5)$$

First we let  $t = t_k$  and  $t' > 0$  to be chosen later, then we let  $t_k \rightarrow 0^+$  so that  $u(t_k) \rightharpoonup \mu$ , and we get

$$\left[ \int_{\mathbb{R}^d} \varphi \, d\mu \right]^\alpha \leq \left[ \int_{\mathbb{R}^d} u(t', x) \, dx \right]^\alpha + K_2 \omega(t'). \quad (7.6)$$

Then we put  $t' = t'_k$  and let  $t'_k \rightarrow 0^+$  so that  $u(t'_k) \rightharpoonup \nu$ ,  $\omega(t'_k) \rightarrow \omega(0) = 0$  and we obtain the first inequality

$$\left[ \int_{\mathbb{R}^d} \varphi \, d\mu \right]^\alpha \leq \left[ \int_{\mathbb{R}^d} \varphi \, d\nu \right]^\alpha. \quad (7.7)$$

Then, we proceed exactly in the same way but we exchange the roles of  $t_k$  and  $t'_k$  to obtain the opposite inequality  $\left[ \int_{\mathbb{R}^d} \varphi \, d\mu \right]^\alpha \leq \left[ \int_{\mathbb{R}^d} \varphi \, d\nu \right]^\alpha$ . Therefore we have that  $\mu = \nu$  as positive linear functionals on  $C_c^\infty(\mathbb{R}^d)$  as desired.  $\square$

**Theorem 7.2 (Existence and uniqueness of initial trace, FD case)** *Let  $0 < m < 1$  and let  $u$  be a nonnegative weak solution of equation (1.1) in  $(0, T] \times \mathbb{R}^d$ . Assume that  $\|u(T)\|_{L^1(\mathbb{R}^d)} < \infty$ . Then there exists a unique nonnegative Radon measure  $\mu$  as initial trace, that is*

$$\int_{\mathbb{R}^d} \psi \, d\mu = \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^d} u(t, x) \psi(x) \, dx, \quad \text{for all } \psi \in C_0(\mathbb{R}^d). \quad (7.8)$$

Moreover, the initial trace  $\mu$  satisfies the bound

$$\mu(B_R(x_0)) \leq \|u(T)\|_{L^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s} T. \quad (7.9)$$

where  $C_1 = C_1(m, d, s) > 0$  as in (2.3).

*Proof.* The proof is divided into three steps.

• **STEP 1. Weighted estimates I. Existence.** First we recall the weighted estimates of Theorem 2.2, which imply for all  $0 \leq t \leq T_1 \leq T$

$$\begin{aligned} \left( \int_{\mathbb{R}^d} u(t, x) \phi_R(x) \, dx \right)^{1-m} &\leq \left( \int_{\mathbb{R}^d} u(T, x) \phi_R(x) \, dx \right)^{1-m} + C_1 R^{d(1-m)-2s} |T - T_1| \\ &\leq \|u(T)\|_{L^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s} T := K_1 \end{aligned} \quad (7.10)$$

since  $\phi_R \leq 1$  and where  $C_1 > 0$  depends only on  $\alpha, m, d$  as in Theorem 2.2. Since  $\phi_R \geq 1$  on  $B_R$  it is clear that this implies hypothesis (i) of Lemma 7.1, therefore it guarantees the existence of an initial trace that satisfies the bound  $\mu(B_R(x_0)) \leq K_1 = \|u(T)\|_{L^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s} T$ .

• **STEP 2. Pseudo-local estimates. Uniqueness.** In order to prove uniqueness of the initial trace is sufficient to prove hypothesis (ii) of Lemma 7.1, namely we need to prove that

$$\left[ \int_{\mathbb{R}^d} u(t, x) \psi(x) \, dx \right]^\alpha \leq \left[ \int_{\mathbb{R}^d} u(t', x) \psi(x) \, dx \right]^\alpha + K_2 \omega(|t - t'|) \quad (7.11)$$

for all  $0 < t, t' \leq T_1 \leq T$  and for all  $\psi \in C_c^\infty(\mathbb{R}^d)$ . We will see that this is true for  $\alpha = 1$  and  $\omega(|t - t'|) = |t - t'|$ . Let  $\psi \in C_c^\infty(\mathbb{R}^d)$ , then we have

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) \psi(x) \, dx \right| &= \left| \int_{\mathbb{R}^d} (-\Delta)^s u^m \psi \, dx \right| \stackrel{(a)}{=} \left| \int_{\mathbb{R}^d} u^m (-\Delta)^s \psi \, dx \right| \leq \int_{\mathbb{R}^d} u^m \phi_R(x) \frac{|(-\Delta)^s \psi(x)|}{\phi_R(x)} \, dx \\ &\stackrel{(b)}{\leq} \left\| \frac{|(-\Delta)^s \psi(x)|}{\phi_R(x)} \right\|_{L^\infty(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} \phi_R \, dx \right)^{1-m} \left( \int_{\mathbb{R}^d} u \phi_R \, dx \right)^m \\ &\leq k_7 \|\phi_R\|_{L^1(\mathbb{R}^d)} K_1 := K_2. \end{aligned}$$

Notice that in (a) we have used the fact that  $(-\Delta)^s$  is a symmetric operator. In (b) we have chosen  $\phi_R(x) := \phi(x/R)$ , with  $\phi$  as in (2.2) of Lemma 2.1, with the decay at infinity  $\alpha = d + 2s$ . It then follows that

$$\left\| \frac{|(-\Delta)^s \psi(x)|}{\phi_R(x)} \right\|_{L^\infty(\mathbb{R}^d)} \leq k_7,$$

since we know by Lemma 2.1 that  $|(-\Delta)^s \psi(x)| \leq k_5 |x|^{-(d+2s)}$ , and we have chosen  $\phi_R \geq k_6/|x|^{d+2s}$ . We have also used the fact that the  $L^m$ -norm ( $m < 1$ ) is less than the  $L^1$  norm since the measure  $\phi_R dx$  is finite. In the last line of the display we have used the bound of Step 1, namely that  $(\int_{\mathbb{R}^d} u(t, x) \phi_R(x) dx)^{1-m} \leq M_T + C_1 R^{d(1-m)-2s} T := K_1$  for all  $0 \leq t \leq T_1$ . Summing up, we have obtained:

$$\left| \frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) \psi(x) dx \right| \leq K_2.$$

Integrating the above differential inequality we obtain:

$$\int_{\mathbb{R}^d} u(t, x) \psi(x) dx \leq \int_{\mathbb{R}^d} u(s, x) \psi(x) dx + K_2 |t - s| \quad \text{for any } s, t \geq 0 \text{ and all } \psi \in C_c^\infty(\mathbb{R}^d). \quad (7.12)$$

• **STEP 3.** We still have to pass from test functions  $\psi \in C_c^\infty(\mathbb{R}^d)$  to  $\psi \in C_c^0(\mathbb{R}^d)$  in formula (7.8), but this is easy by approximation (mollification).  $\square$

**Remarks.** (i) The proof applies with minor modification to the class of solutions with data  $u_0 \in L^1(\mathbb{R}^d, \varphi dx)$  constructed in Section 3.

(ii) Notice that estimates (7.12) are only pseudo-local estimates: the global information about  $u(T)$ , namely the bound  $\|u(T)\|_{L^1(\mathbb{R}^d)}$  is contained in the constant  $K_1$  and therefore in  $K_2$ .

(iii) The existence of solutions and traces for the standard FDE with (not necessarily locally finite) Borel measures as data is studied in Chasseigne-Vazquez [7]. We do not address the corresponding question here.

**Theorem 7.3 (Existence and uniqueness of initial trace, PME case)** *Let  $m > 1$  and let  $u$  be a solution to the Cauchy problem 1.1 on  $(0, T] \times \mathbb{R}^d$ . Assume that  $\|u(T)\|_{L^1(\mathbb{R}^d)} + \|u(T)\|_{L^\infty(\mathbb{R}^d)} < +\infty$ . Then there exists a unique nonnegative Borel measure  $\mu$  as initial trace, that is*

$$\int_{\mathbb{R}^d} \psi d\mu = \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^d} u(t, x) \psi(x) dx, \quad \text{for all } \psi \in C_0(\mathbb{R}^d). \quad (7.13)$$

Moreover the initial trace  $\mu$  satisfies the bound

$$\mu(B_R(x_0)) \leq C_1 \left[ \left( \frac{R^{2s+d(m-1)}}{T} \right)^{\frac{1}{m-1}} + T^{\frac{d}{2s}} u(x_0, T)^{\frac{1}{2s\vartheta}} \right], \quad (7.14)$$

where  $C_1 = C_1(m, d, s) > 0$  as in Theorem 6.1.

*Proof.* The proof is divided in three steps

• **STEP 1. Weighted estimates I. Existence.** First we recall the lower bounds of Theorem (6.1) rewritten in the form (6.3)

$$\int_{B_R(x_0)} u(s, x) dx \leq C_1 \left[ \left( \frac{R^{2s+d(m-1)}}{T} \right)^{\frac{1}{m-1}} + T^{\frac{d}{2s}} u(x_0, T)^{\frac{1}{2s\vartheta}} \right] := K_1. \quad (7.15)$$

on the time interval  $(s, T] \subseteq (0, T]$ . It is clear that this implies hypothesis (i) of Lemma 7.1, therefore it guarantees the existence of an initial trace that satisfy the bound  $\mu(B_R(x_0)) \leq K_1$ .

• **STEP 2. Smoothing effects and mass conservation.** In [12] there are the global  $L^1 - L^\infty$  smoothing effects which provide global upper bounds for solutions to the Cauchy problem 1.1. We apply such smoothing effects to solutions to our reduced Problem 4.5 to get

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{2^{d\vartheta} I_\infty}{t^{d\vartheta}} \|u(t/2)\|_{L^1(\mathbb{R}^d)}^{2s\vartheta} \quad (7.16)$$

where  $\vartheta = 1/[2s + d(m-1)]$  and the constant  $I_\infty$  only depends on  $d, s, m$ . Moreover, we know that there holds also the conservation of mass on the time interval  $[t/2, T] \subset (0, T]$ , so that inequality (7.16) becomes

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{2^{d\vartheta} I_\infty}{t^{d\vartheta}} \|u(T)\|_{L^1(\mathbb{R}^d)}^{2s\vartheta}. \quad (7.17)$$

• **STEP 3. Weighted estimates II. Pseudo-local estimates. Uniqueness.** In order to prove uniqueness of the initial trace is sufficient to prove hypothesis (ii) of Lemma 7.1, namely we need to prove

$$\left[ \int_{\mathbb{R}^d} u(t, x) \psi(x) dx \right]^\alpha \leq \left[ \int_{\mathbb{R}^d} u(t', x) \psi(x) dx \right]^\alpha + K_2 \omega(|t - t'|) \quad (7.18)$$

for all  $0 < t, t' \leq T_1 \leq T$  and for all  $\psi \in C_c^\infty(\mathbb{R}^d)$ . We will see that this is true for  $\alpha = 1$  and  $\omega(|t - t'|) = |t^\sigma - t'^\sigma|$  with  $\sigma = 2s/[2s + d(m-1)]$ . Let  $\psi \in C_c^\infty(\mathbb{R}^d)$ , then we have

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) \psi(x) dx \right| &= \left| \int_{\mathbb{R}^d} (-\Delta)^s u^m \psi dx \right| =_{(a)} \left| \int_{\mathbb{R}^d} u^m (-\Delta)^s \psi dx \right| \\ &\leq \|u(t)\|_{L^\infty(\mathbb{R}^d)}^{m-1} \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} u(t) dx \\ &\leq_{(b)} \frac{2^{d\vartheta(m-1)} I_\infty^{m-1}}{t^{d\vartheta(m-1)}} \|u(T)\|_{L^1(\mathbb{R}^d)}^{2s\vartheta(m-1)} \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^d)} \|u(t)\|_{L^\infty(\mathbb{R}^d)} \\ &= \frac{2^{d\vartheta(m-1)} I_\infty^{m-1}}{t^{d\vartheta(m-1)}} \|u(T)\|_{L^1(\mathbb{R}^d)}^{2s\vartheta(m-1)+1} \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^d)} := \frac{K_2}{t^{d\vartheta(m-1)}}. \end{aligned}$$

Notice that in (a) we have used the fact that  $(-\Delta)^s$  is a symmetric operator. In (b) we have used the smoothing effect (7.17) of Step 2. Summing up we have obtained:

$$\left| \frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) \psi(x) dx \right| \leq \frac{K_2}{t^{d\vartheta(m-1)}}.$$

Integrating the above differential inequality we obtain for any  $s, t \geq 0$ :

$$\int_{\mathbb{R}^d} u(t, x) \psi(x) dx \leq \int_{\mathbb{R}^d} u(s, x) \psi(x) dx + 2s\vartheta K_2 \left| t^{2s\vartheta} - s^{2s\vartheta} \right| \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^d). \quad (7.19)$$

• **STEP 3.** We still have to pass from test functions  $\psi \in C_c^\infty(\mathbb{R}^d)$  to  $\psi \in C_c^0(\mathbb{R}^d)$  in formula (7.13), but this is easy by approximation (mollification).  $\square$

We notice that the estimates (7.19) are only pseudo-local estimates: the global information about  $u(T)$ , namely the bound  $\|u(T)\|_{L^1(\mathbb{R}^d)}$  is contained in the constant  $K_2$ .

## 8 Appendix: Complements and computations

### 8.1 Definition of the fractional Laplacian.

According to Stein, [15], chapter V, the definition of the nonlocal operator  $(-\Delta)^{\sigma/2}$ , known as the Laplacian of order  $\sigma$ , is done by means of Fourier series

$$((-\Delta)^{\sigma/2}f)\widehat{f}(x) = (2\pi|x|)^{\sigma}\widehat{f}(x), \quad (8.1)$$

and can be used for positive and negative values of  $\sigma$ . If  $0 < \sigma < 2$ , we can also use the representation by means of an hypersingular kernel,

$$(-\Delta)^{\sigma/2}g(x) = c_{d,\sigma} \text{P.V.} \int_{\mathbb{R}^d} \frac{g(x) - g(z)}{|x - z|^{d+\sigma}} dz, \quad (8.2)$$

where  $c_{d,\sigma} = \frac{2^{\sigma-1}\sigma\Gamma((d+\sigma)/2)}{\pi^{d/2}\Gamma(1-\sigma/2)}$  is a normalization constant. Another classical way of defining the fractional powers of a linear self-adjoint nonnegative operator, in terms of the associated semigroup, which in our case reads

$$(-\Delta)^{\sigma/2}g(x) = \frac{1}{\Gamma(-\frac{\sigma}{2})} \int_0^\infty (e^{t\Delta}g(x) - g(x)) \frac{dt}{t^{1+\frac{\sigma}{2}}}. \quad (8.3)$$

In this paper we consistently put  $\sigma = 2s$ ,  $0 < s < 1$  (sometimes, also  $s = 1$ ).

### 8.2 Definition of weak and very weak solutions

We recall here the definitions of weak and strong solutions taken from [12]. We finally introduce the definition of very weak solutions.

**Definition 8.1** *A function  $u$  is a weak solution to Equation (1.1) if:*

- $u \in C((0, \infty) : L^1(\mathbb{R}^d))$ ,  $|u|^{m-1}u \in L_{\text{loc}}^2((0, \infty) : \dot{H}^s(\mathbb{R}^d))$ ;
- *The identity*

$$\int_0^\infty \int_{\mathbb{R}^d} u \frac{\partial \varphi}{\partial t} dx ds - \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{s/2}(|u|^{m-1}u)(-\Delta)^{s/2}\varphi dx ds = 0. \quad (8.4)$$

*holds for every  $\varphi \in C_0^1(\mathbb{R}^d \times (0, \infty))$ ;*

- *A weak solution to Problem (1.1)–(1.2) is a weak solution to Equation (1.1) such that moreover  $u \in C([0, \infty) : L^1(\mathbb{R}^d))$  and  $u(0, \cdot) = u_0 \in L^1(\mathbb{R}^d)$ .*

Note that in [12] these weak solutions are given the more precise name *weak  $L^1$ -energy solutions*. We recall that the fractional Sobolev space  $\dot{H}^s(\mathbb{R}^d)$  is defined as the completion of  $C_0^\infty(\mathbb{R}^d)$  with the norm

$$\|\psi\|_{\dot{H}^s} = \left( \int_{\mathbb{R}^d} |\xi|^\sigma |\hat{\psi}|^2 d\xi \right)^{1/2} = \|(-\Delta)^{s/2}\psi\|_2.$$

**Definition 8.2** We say that a weak solution  $u$  to Problem (1.1)–(1.2) is a strong solution if moreover  $\partial_t u \in L^\infty((\tau, \infty) : L^1(\mathbb{R}^d))$ , for every  $\tau > 0$ .

**Definition 8.3** A function  $u$  is a very weak solution to Equation (1.1) if:

- $u \in C((0, \infty) : L^1_{\text{loc}}(\mathbb{R}^d))$ ,  $|u|^{m-1}u \in L^1_{\text{loc}}((0, \infty) : L^1(\mathbb{R}^d, (1 + |x|)^{-(d+2s)} dx))$ ;
- The identity

$$\int_0^\infty \int_{\mathbb{R}^d} u \frac{\partial \varphi}{\partial t} dx ds - \int_0^\infty \int_{\mathbb{R}^d} |u|^{m-1} u (-\Delta)^s \varphi dx ds = 0. \quad (8.5)$$

holds for every  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ ;

- A very weak solution to Problem (1.1)–(1.2) is very weak solution to Equation (1.1) such that moreover  $u \in C([0, \infty) : L^1_{\text{loc}}(\mathbb{R}^d))$  and  $u(0, \cdot) = u_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$ .

### 8.3 Some functional inequalities related to the fractional Laplacian

We recall here some useful functional inequalities which have been used throughout the paper.

**Lemma 8.4 (Stroock-Varopoulos' inequality)** Let  $0 < s < 1$ ,  $q > 1$ . Then

$$\int_{\mathbb{R}^d} |v|^{q-2} v (-\Delta)^s v dx \geq \frac{4(q-1)}{q^2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} |v|^{\frac{q}{2}} \right|^s dx, \quad (8.6)$$

for all  $v \in L^q(\mathbb{R}^d)$  such that  $(-\Delta)^s v \in L^q(\mathbb{R}^d)$ .

**Remark.** We have used the above Stroock-Varopoulos inequality, applied to  $0 \leq v = u^m$  and  $q = (p + m - 1)/m > 1$ , whenever  $p > 1$ , which is

$$\int_{\mathbb{R}^d} |u|^{p-2} u (-\Delta)^s (|u|^{m-1} u) dx \geq \frac{4m(p-1)}{(p+m-1)^2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} |u|^{\frac{p+m-1}{2}} \right|^s dx. \quad (8.7)$$

**Theorem 8.5 (Sobolev Inequality)** Let  $0 < s \leq 1$  and  $2s < d$ . Then

$$\|f\|_{\frac{2d}{d-2s}} \leq \mathcal{S}_s \left\| (-\Delta)^{s/2} f \right\|_2 \quad (8.8)$$

where the best constant is given by

$$\mathcal{S}_s^2 := 2^{-2s} \pi^{-s} \frac{\Gamma(\frac{d-2s}{2})}{\Gamma(\frac{d+2s}{2})} \left[ \frac{\Gamma(d)}{\Gamma(d/2)} \right]^{\frac{2s}{d}} = \frac{\Gamma(\frac{d-2s}{2})}{\Gamma(\frac{d+2s}{2})} |\mathbb{S}_d|^{-\frac{2s}{d}} \quad (8.9)$$

and is attained on the family of functions

$$F(x) := a \left[ b^2 + (x - x_0)^2 \right]^{-\frac{d-2s}{2}}, \quad \text{with } x, x_0 \in \mathbb{R}^d \text{ and } a \in \mathbb{R}, b > 0.$$

## 8.4 Proof of Lemma 2.1

*Proof.* The proof is divided into several steps.

- STEP 1. *The integral is convergent.* First we have to prove that

$$c_{d,s}^{-1} |(-\Delta)^s \varphi(x)| = \left| \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy \right| < \infty \quad \text{for any } x \in \mathbb{R}^d$$

to this end we fix  $x \in \mathbb{R}^d$  and we split the integral in two parts:

$$\int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy = \int_{|x-y|>\delta} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy + \int_{|x-y|\leq\delta} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy = I + II$$

where  $\delta > 0$  is taken so small that the following Taylor expansion around  $x \in \mathbb{R}^d$  holds true

$$\varphi(y) = \varphi(x) + \nabla \varphi(x) \cdot (y - x) + (y - x)^t D^2 \varphi(\bar{x}) (y - x)$$

for some  $\bar{x} \in B_1(x)$ . Therefore we have

$$\begin{aligned} I &= \left| \int_{|x-y|\leq\delta} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy \right| \\ &= \left| \int_{|x-y|\leq\delta} \frac{\nabla \varphi(x) \cdot (y - x)}{|x - y|^{d+2s}} dy + \int_{|x-y|\leq\delta} \frac{(y - x)^t D^2 \varphi(\bar{x}) (y - x)}{|x - y|^{d+2s}} dy \right| \\ &\leq_{(a)} \sup_{1 \leq i, j \leq d} \|\partial_{ij} \varphi\|_{L^\infty(\mathbb{R}^d)} \left| \int_{|x-y|\leq\delta} \frac{1}{|x - y|^{d-(2-2s)}} dy \right| \\ &\leq \sup_{1 \leq i, j \leq d} \|\partial_{ij} \varphi\|_{L^\infty(\mathbb{R}^d)} \int_0^\delta \frac{dr}{r^{1-2(1-s)}} \stackrel{(b)}{=} K \frac{\delta^{2(1-s)}}{(2(1-s))} \end{aligned}$$

where in (a) we have used that

$$P.V. \int_{|x-y|\leq\delta} \frac{\nabla \varphi(x) \cdot (y - x)}{|x - y|^{d+2s}} dy = 0$$

for symmetry reasons. In (b) we used the fact that  $|\partial_{ij} \varphi(z)| \leq K$  for some positive constant  $K$  that depends only on  $\alpha$ . On the other hand, the outer integral is easily seen to be finite, indeed

$$II = \left| \int_{|x-y|>\delta} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy \right| \leq 2 \|\varphi\|_{L^\infty(\mathbb{R}^d)} \left| \int_{|x-y|>\delta} \frac{1}{|x - y|^{d+2s}} dy \right| \leq 2\omega_d \int_\delta^\infty \frac{dr}{r^{1+2s}} = \frac{\omega_d}{s\delta^{2s}}.$$

The above estimates for  $I$  and  $II$  do not depend on  $x \in \mathbb{R}^d$ , hence  $|(-\Delta)^s \varphi(x)|$  is finite for all  $x \in \mathbb{R}^d$ .

- STEP 2. *Better estimates for  $|x|$  large.* We are going to use the hypothesis that  $\varphi$  is radially symmetric and decreasing for  $|x| \geq 1$  and that  $\varphi(x) \leq |x|^{-\alpha}$ ,  $|D^2 \varphi(x)| \leq c_0 |x|^{-\alpha-2}$ , for some positive



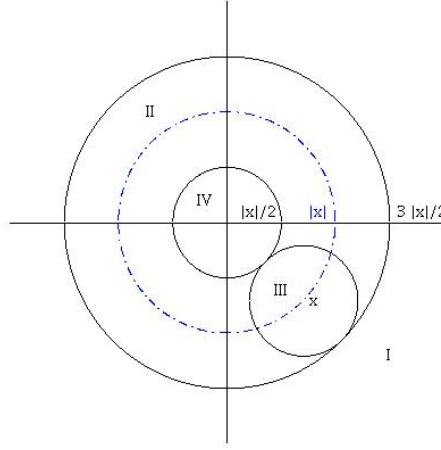


Figure 3: *The 4 regions in which we split the integral*

constant  $\alpha$  and for  $|x|$  large enough. We are interested in the behaviour of  $|(-\Delta)^s \varphi(x)|$  for large values of  $x$ , therefore we fix  $x \in \mathbb{R}^d$  with  $|x|$  sufficiently large. We have to estimate

$$c_{d,s}^{-1} |(-\Delta)^s \varphi(x)| = \left| \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy \right|,$$

to this end we split the integral into four parts, see Figure 3,

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy &= \int_{|y| > 3|x|/2} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy + \int_{\{|x| \leq 2|y| \leq 3|x|\} \setminus B_{|x|/2}(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy \\ &+ \int_{B_{|x|/2}(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy + \int_{|y| < |x|/2} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy = I + II + III + IV \end{aligned} \quad (8.10)$$

We estimate the four integrals separately, keeping in mind that we are assuming  $\varphi \geq 0$  in this latter case. The first integral can be estimated as follows

$$I = \int_{|y| > 3|x|/2} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy \leq \omega_d \varphi(x) \int_{\frac{3|x|}{2}}^{\infty} \frac{dr}{r^{1+2s}} = \frac{k_1}{|x|^{\alpha+2s}}$$

since  $\varphi(y) \leq \varphi(x)$  when  $|y| > 3|x|/2$ , therefore  $|\varphi(x) - \varphi(y)| \leq \varphi(x)$ , and we remark that the constant  $k_1$  depends only on  $\alpha, s, d$ , since  $\varphi(x) \leq |x|^{-\alpha}$  and  $|x|$  is large enough. The second integral gives

$$II \leq \left| \int_{\{|x| \leq 2|y| \leq 3|x|\} \setminus B_{|x|/2}(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2s}} dy \right| \leq \frac{\varphi(x/2)}{(|x|/2)^{d+2s}} \int_{\frac{|x|}{2}}^{\frac{3|x|}{2}} r^{d-1} dr \leq \frac{k_2}{|x|^{\alpha+2s}}$$

since  $\varphi(y) \leq \varphi(x/2)$  when  $|y| > |x|/2$ , therefore  $|\varphi(x) - \varphi(y)| \leq \varphi(x/2)$ , and we remark that the constant  $k_2$  depends only on  $\alpha, s, d$ , since  $\varphi(x/2) \leq |x|^{-\alpha}$  and  $|x|$  is large enough.

We can estimate the third integral as follows:

$$\begin{aligned}
III &= \left| \int_{|x-y| \leq \frac{|x|}{2}} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+2s}} dy \right| \\
&= \left| \int_{|x-y| \leq \frac{|x|}{2}} \frac{\nabla \varphi(x) \cdot (y-x)}{|x-y|^{d+2s}} dy + \int_{|x-y| \leq \frac{|x|}{2}} \frac{(y-x)^t \text{Hess} \varphi(\bar{x}) (y-x)}{|x-y|^{d+2s}} dy \right| \\
&\leq_{(a)} \sup_{1 \leq i,j \leq d} \|\partial_{ij} \varphi\|_{L^\infty(B_{\frac{|x|}{2}}(x))} \left| \int_{|x-y| \leq \frac{|x|}{2}} \frac{1}{|x-y|^{d-(2-2s)}} dy \right| \\
&\leq \frac{k'_3}{|x|^{\alpha+2}} \int_0^{\frac{|x|}{2}} \frac{dr}{r^{1-2(1-s)}} = \frac{k'_3}{|x|^{\alpha+2}} \left( \frac{|x|}{2} \right)^{2-2s} = \frac{k_3}{|x|^{\alpha+2s}}
\end{aligned}$$

where in (a) we have used that

$$P.V. \int_{|x-y| \leq \delta} \frac{\nabla \varphi(x) \cdot (y-x)}{|x-y|^{d+2s}} dy = 0$$

for symmetry reasons as in Step 2. In (b) we used the fact that  $|z-x| < |x|/2$  implies  $|x|/2 < |z| < 3|x|/2$ , therefore  $|\partial_{ij}^2 \varphi(z)| \leq c_0/|z|^{\alpha+2} \leq 2^{\alpha+2} c_0/|x|^{\alpha+2}$  for all  $z \in B_{|x|/2}(x)$ , recalling that  $|x|$  is always taken large enough. The constants  $k'_3$  and  $k_3$  depend only on  $\alpha, s, d$ .

It only remains to estimate the fourth integral:

$$IV \leq \int_{|y| < |x|/2} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{d+2s}} dy \leq \frac{2^{d+2s}}{|x|^{d+2s}} \int_{|y| < |x|/2} \varphi(y) dy.$$

since we observe that  $|y| < |x|/2$  implies  $\varphi(x) \leq \varphi(2y) \leq \varphi(y)$  which gives  $|\varphi(x) - \varphi(y)| \leq \varphi(y)$ , moreover we also have that  $|y| < |x|/2$  implies that  $|y-x| > |x|/2$ . The term represents the long-range influence of the inner core of the function at large distances and will make for different conclusions of the lemma depending on the case. Indeed, we have the following estimates for  $|x|$  large enough:

- If  $\alpha > d$  the last integral is finite and we get  $IV \leq k_4/|x|^{d+2s}$ .
- If  $\alpha < d$  the last integral grows like  $|x|^{d-\alpha}$  and we get  $IV \leq k_5/|x|^{\alpha+2s}$ .
- Finally when  $\alpha = d$  we get  $IV \leq k_6 \log |x|/|x|^{d+2s}$ .

We finally remark that the constants  $k_4, k_5, k_6$  depend only on  $\alpha, s, d$ .

• **STEP 3. Positivity estimates for  $|x|$  large.** In the case when  $\alpha > d$  we need to prove that if  $\varphi \geq 0$  then we have that  $|(-\Delta)^s \varphi(x)| \geq c_4 |x|^{-(d+2s)}$  for all  $|x| \geq |x_0| \gg 1$ . We split the integral into four parts, as in Step 2, equation (8.10), see Figure 3,

$$c_{d,s}^{-1}(-\Delta)^s \varphi(x) = \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+2s}} dy = I + II + III + IV$$

We have proven in Step 2 that  $|I| + |II| + |III| \leq (k_1 + k_2 + k_3)/|x|^{\alpha+2s}$  and we recall that the constant  $k_i$  depend only on  $\alpha, s, d$ . We just have to obtain better estimates for the last term, to this end we further split the integral in two parts:

$$IV = \int_{|y| < |x|/2} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+2s}} dy = \int_{|y| < |x|/2} \frac{\varphi(x)}{|x-y|^{d+2s}} dy - \int_{|y| < |x|/2} \frac{\varphi(y)}{|x-y|^{d+2s}} dy = IV_a - IV_b$$

Let us calculate

$$0 \leq IV_a = \int_{|y| < |x|/2} \frac{\varphi(x)}{|x-y|^{d+2s}} dy \leq \frac{\varphi(x)}{(|x|/2)^{d+2s}} \int_{|y| < |x|/2} dy \leq \frac{k_4}{|x|^{\alpha+2s}}$$

since  $|x-y| \geq |x|/2$  when  $|y| \leq |x|/2$  and  $\varphi(x) \leq |x|^{-\alpha}$ . We remark that the constant  $k_4$  depends only on  $\alpha, s, d$ . On the other hand,  $IV_b \geq 0$  and

$$0 \leq IV_b = \int_{|y| < |x|/2} \frac{\varphi(y)}{|x-y|^{d+2s}} dy \leq \frac{1}{(|x|/2)^{d+2s}} \int_{|y| < |x|/2} \varphi(y) dy \leq \frac{\|\varphi\|_{L^1(\mathbb{R}^d)}}{|x|^{d+2s}}$$

Summing up, we have obtained that

$$\begin{aligned} -(-\Delta)^s \varphi(x) &= -c_{d,s} \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+2s}} dy \geq c_{d,s} [IV_b - (|I| + |II| + |III| + |IV_a|)] \\ &\geq c_{d,s} \left[ \frac{\|\varphi\|_{L^1(\mathbb{R}^d)}}{|x|^{d+2s}} - \frac{k_5}{|x|^{\alpha+2s}} \right] = \left[ \|\varphi\|_{L^1(\mathbb{R}^d)} - \frac{k_5}{|x|^{\alpha-d}} \right] \frac{c_{d,s}}{|x|^{d+2s}} \geq \frac{c_4}{|x|^{d+2s}} \end{aligned}$$

since  $|I| + |II| + |III| + |IV_a| \leq (k_1 + k_2 + k_3 + k_4) |x|^{\alpha+2s} = k_5 |x|^{\alpha+2s}$  and  $c_4 > 0$  since  $\alpha > d$ , if we choose  $|x|$  sufficiently large, namely  $|x|^{\alpha-d} \geq k_5 / \|\varphi\|_{L^1(\mathbb{R}^d)}$ .  $\square$

## 8.5 Optimization Lemma

We state and prove here a simple technical lemma that has been used in the proof of Theorem 4.1.

**Lemma 8.6** *Let  $0 < m, s < 1$ ,  $2s > d(1-m)$ ,  $\vartheta = 1/[2s - d(1-m)] > 0$  and  $B, C, t > 0$ . Define*

$$F(t, R) := \frac{A(t)}{R^{d(1-m)}} - \frac{Bt}{R^{2s}}, \quad \text{with} \quad A(t) := M - \frac{C}{t^{d(1-m)\vartheta}}.$$

*Then there exists*

$$t_* := 2s\vartheta \left( \frac{C}{M} \right)^{\frac{1}{d(1-m)\vartheta}} > 0 \quad (8.11)$$

*and*

$$\bar{R}(t) = \left( \frac{2sBt}{d(1-m)A(t)} \right)^{\vartheta} \geq \bar{R}(t_*) = \left[ \frac{2s}{d(1-m)} \frac{(2s\vartheta)^{2s\vartheta}}{(2s\vartheta)^{d(1-m)} - 1} \right]^{\vartheta} \frac{B^{\vartheta} C^{\frac{1}{d(1-m)}}}{M^{\frac{2s\vartheta}{d(1-m)}}} > 0 \quad (8.12)$$

*so that for all  $t \geq t_*$  we have*

$$F(\bar{R}(t), t) = \left[ \left( \frac{2s}{d(1-m)} \right)^{\frac{1}{\vartheta}} - 1 \right] \left[ \frac{d(1-m)}{2s} \right]^{2s\vartheta} \frac{A(t)^{2s\vartheta}}{(Bt)^{d(1-m)\vartheta}} > 0.$$

*Proof.* First we observe that  $A(t)$  is monotone increasing in  $t > 0$ , and that  $A(t_*/2s\vartheta) = 0$ , where  $t_*$  has the expression given by (8.11), so that  $A(t) > A(t_*) > A(t/2s\vartheta) = 0$  since  $2s\vartheta > 1$ , and

$$A(t_*) = \frac{(2s\vartheta)^{d(1-m)\vartheta} - 1}{(2s\vartheta)^{d(1-m)\vartheta}} M > 0$$

Moreover, it is easy to check that  $t_*$  is also the value for which  $A(t_*) - t_* A'(t_*) = 0$ . Next we fix a time  $t \geq t_*$  and we find the maximum with respect to  $R$  of the function  $F(t, R)$ :

$$\partial_R F(R, t) = -\frac{d(1-m)A(t)}{R^{d(1-m)\vartheta+1}} + \frac{2sBt}{R^{2s+1}},$$

and  $\partial_R F(\bar{R}(t), t) = 0$ , so that the maximum is attained at  $\bar{R}(t)$  whose expression it is easily checked to be (8.12). It only remains to prove that  $\bar{R}(t) \geq \bar{R}(t_*) > 0$ , to this end we observe that

$$\partial_t \bar{R}(t) = \vartheta \left[ \frac{2sB}{d(1-m)} \right]^\vartheta \left[ \frac{t}{A(t)} \right]^{\vartheta-1} \frac{A(t) - tA'(t)}{A(t)^2}$$

and it is clear now that the minimum is attained at  $t_*$ , since  $\partial_t \bar{R}(t_*) = 0$ , because we already know that  $A(t_*) - t_* A'(t_*) = 0$ .  $\square$

## 8.6 Reminder about measure theory

We recall here some basic facts on measure theory for convenience of the reader. We refer the interested reader to the books [9, 14].

**Definition 8.7** *A measure  $\mu$  is regular if*

$$\forall A \subseteq \mathbb{R}^d \exists B \text{ } \mu\text{-measurable such that } A \subseteq B \text{ and } \mu(A) = \mu(B).$$

*A measure  $\mu$  is Borel if every Borel set  $\mathcal{B}(\mathbb{R}^d)$  is  $\mu$ -measurable. A measure  $\mu$  is Borel regular if*

$$\forall A \subseteq \mathbb{R}^d \exists B \in \mathcal{B}(\mathbb{R}^d) \text{ such that } A \subseteq B \text{ and } \mu(A) = \mu(B).$$

*A measure  $\mu$  is Radon if it is Borel regular and  $\mu(K) < +\infty$  for any compact set  $K \subset \mathbb{R}^d$ .*

*A sequence of measures  $\mu_n$  converges weakly (star) to the measure  $\mu$ ,  $\mu_n \rightharpoonup \mu$  as  $n \rightarrow \infty$  if*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi d\mu_n = \int_{\mathbb{R}^d} \varphi d\mu \quad \text{for all } \varphi \in C_c^0(\mathbb{R}^d).$$

**Theorem 8.8 (Weak compactness for measures)** *Let  $\{\mu_n\}$  be a sequence of Radon measures on  $\mathbb{R}^d$  satisfying*

$$\sup_n \mu_n(K) < \infty \quad \text{for any compact set } K \subset \mathbb{R}^d.$$

*Then there exists a subsequence  $\mu_{n_k}$  and a Radon measure  $\mu$  such that  $\mu_{n_k} \rightharpoonup \mu$  as  $k \rightarrow \infty$ .*

**Theorem 8.9 (Riesz Representation Theorem)** *Assume  $L : C_c^\infty(\mathbb{R}^d) \rightarrow \mathbb{R}$  is linear and nonnegative, so that*

$$L\varphi \geq 0 \quad \text{for all } 0 \leq \varphi \in C_c^\infty$$

*Then there is a unique Radon measure  $\mu$  on  $\mathbb{R}^d$  such that*

$$L\varphi = \int_{\mathbb{R}^d} \varphi d\mu \quad \text{for all } \varphi \in C_c^\infty$$

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